Exponential Barrier Method in Solving Linear Programming Problems

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***Abstract*—This paper is concerned with the study of the exponential barrier method for linear programming problems with the essential property that each exponential barrier method is concave when viewed as a function of the multiplier.. It presents some background of the method and its variants for the problem. Under certain assumption on the parameters of the exponential barrier function, we give a rule for choosing the parameters of the barrier function. Theorems and algorithms for the methods are also given in this paper. At the end of the paper we give some conclusions and comments on the methods.**

***Index Terms*—linear programming, exponential barrier method, barrier function. Interior pint algorithm.**

I. INTRODUCTION

The basic idea in exponential barrier method is to eliminate some or all of the constraints and add to the objective function a barrier term which prescribes a high cost to infeasible points (Wright, 2001). Associated with this method is a parameter σ, which determines the severity of the barrier and as a consequence the extent to which the resulting unconstrained problem approximates the original problem (Parwadi, etc., 2002). Parwadi (2010) proposed a polinomial penalty method for solving linear programming problems. In this paper, we restrict attention to the exponential barrier function. It presents some background of the methods for the problem. The paper also describes the theorems and algorithms for the methods. At the end of the paper we give some conclusions and comments to the methods.

Throughout this paper we consider the problem

$$\begin{aligned} &\text{maximize} \quad c^T x \\ &\text{subject to } Ax = b \\ &\qquad\qquad x \geq 0, \end{aligned} \tag{1}$$

where $A \in R^{m\times n}$, $c, x \in R^n$, and $b \in R^m$. Without loss of generality we assume that A has full rank m. We assume that problem (1) has at least one feasible solution. In order to solve this problem, we can use Karmarkar's algorithm and simpelx method (Durazzi, 2000).

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Parwadi (2010 and 2011) also has introduced a polynomial penalty and barrier methods for solving primal-dual linear programming problems. But in this paper we propose an exponential barrier method as another alternative method to solve linear programming problem (1).

II. THE EXPONENTIAL BARRIER FUNCTION

We consider the linear programming stated in (1). The exponential barrier function is given by

$$E(x,\sigma) = c^T x - \sigma \sum_{i=1}^{m} \exp\{\sigma(A_i x - b_i)\} \tag{2}$$

where σ > 0 is a barrier parameter of the function. The barrier is formed from a sum of exponential of constraint violations and the parameter σ determines the amount of the barrier. The basic idea in this method is to eliminate all constraints and add to the objective function a barrier term which prescribes a high cost to infeasible points. Associated with this function is a barrier parameter σ, which determines the severity of the barrier and as a consequence the extent to which the resulting unconstrained problem approximates the original constrained problem.

In order to understand the behavior of this function we give the trivial problem

$$\begin{aligned} &\text{maximize} \quad f(x) = x \\ &\text{subject to} \quad x - 1 = 0, \end{aligned}$$

for which the exponential barrier function is given by

$$E(x,\sigma) = x - \sigma \exp\{\sigma(x-1)\}.$$

Some graphs of $E(x,\sigma)$ are given in Figure 1. This figure depicts the one-dimensional variation of the exponential barrier function for three values of σ, that is σ = 10, σ = 20 and σ = 40. If the solution $x^* = 1$ is compared with the points which minimize $E(x,\sigma)$, it is clear that x^* is a limit point of the unconstrained maximizers of $E(x,\sigma)$ as $\sigma \to \infty$. The *y*-ordinate of this figure represents $E(x,\sigma)$.

The intuitive motivation for an exponential barrier method is that we seek unconstrained maximizers of $E(x,\sigma)$ for value of σ increasing to infinity. Thus the method of solving the sequence of maximization problem can be suggested.

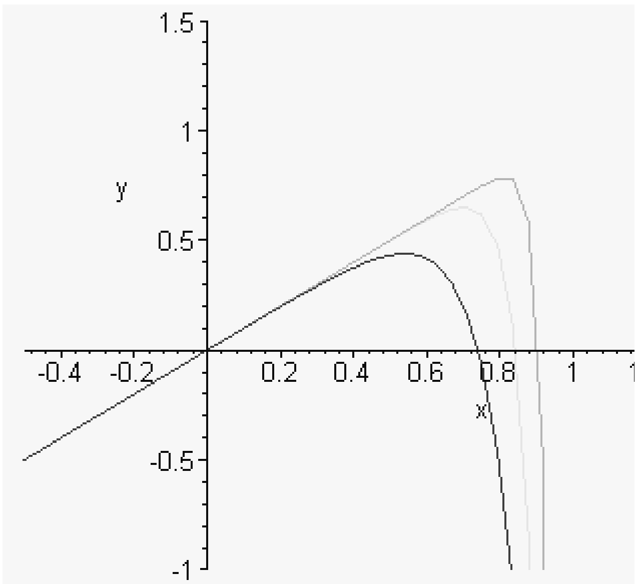

Figure 1. The exponential barrier functions

The exponential barrier method for problem (1) consists of solving a sequence of problems of the form

$$\text{maximize } E(x,\sigma^k) \quad \text{subject to } x \geq 0, \tag{3}$$

where $\{\sigma^k\}$ is a barrier parameter sequence satisfying

$0 < \sigma^k < \sigma^{k+1}$ for all k, and $\sigma^k \to \infty$.

The method depends on the success for sequentially increasing the barrier parameter to infinity. This paper concentrates on the effect of the barrier parameter. It is easy to show that $E(x,\sigma)$ is a concave function for each σ. The concavity behavior of the exponential barrier function defined by (2) is stated in the following theorem.

Theorem 1 (Concavity)
The exponential barrier function $E(x,\sigma)$ defined in (2) is concave in its domain for every $\sigma > 0$.

Proof. It is straightforward to prove concavity of $E(x,\sigma)$ by using the concavity of $c^T x$ and $-\sigma\sum_{i=1}^{m}\exp\{\sigma(A_i x - b_i)\}$. Then this theorem is proven. ■

As a consequence of Theorem 1 we derive the local and global behavior of the exponential barrier function defined by (2) which is stated in the following theorem.

Theorem 2 (Local and global behavior)
(a) $E(x,\sigma)$ has a finite unconstrained maximizer in its domain for every $\sigma > 0$ and the set M_σ of unconstrained maximizers of $E(x,\sigma)$ in its domain is concave and compact for every $\sigma > 0$.
(b) Any unconstrained local maximizer of $E(x,\sigma)$ in its domain is also a global unconstrained maximizer of $E(x,\sigma)$.

Proof. It follows from Theorem 1 that the smooth function $E(x,\sigma)$ achieves its maximum in its domain. We then conclude that $E(x,\sigma)$ has at least one finite unconstrained maximizer.
By Theorem 1 $E(x,\sigma)$ is concave, so any local maximizer is also a global maximizer. Thus the set M_σ of unconstrained maximizers of $E(x,\sigma)$ is bounded and closed, because the maximum value of $E(x,\sigma)$ is unique, and it follows that M_σ is compact. Clearly, the concavity of M_σ follows from the fact that set of optimal points $E(x,\sigma)$ is concave. Theorem 2 is verified. ■

Using the results of Theorem 2 we derive the monotonicity behavior of the maximum value of the exponential barrier function $E(x,\sigma)$. To do this, for any $\sigma^k > 0$ we denote x^k and $E(x^k,\sigma^k)$ as a maximizer and maximum value of the problem (3), respectively.

Theorem 3 (Monotonicity)
Let $\{\sigma^k\}$ be an increasing sequence of positive barrier parameters such that $\sigma^k \to \infty$ as $k \to \infty$. Then $\{E(x^k,\sigma^k)\}$ is non-decreasing.

Proof. Let x^k and x^{k+1} denote global maximizers of the problem (3) for the barrier parameters σ^k and σ^{k+1}, respectively. By definition of x^k and x^{k+1} as maximizers and $\sigma^k < \sigma^{k+1}$, for sufficiently large *k*, we have

$$c^T x^{k+1} - \sigma^{k+1} \sum_{i=1}^{m} \exp\{\sigma^{k+1}(A_i x^{k+1} - b_i)\} \geq c^T x^k - \sigma^{k+1} \sum_{i=1}^{m} \exp\{\sigma^{k+1}(A_i x^k - b_i)\}, \quad (4a)$$

$$c^T x^k - \sigma^{k+1} \sum_{i=1}^{m} \exp\{\sigma^{k+1}(A_i x^k - b_i)\} \geq c^T x^k - \sigma^k \sum_{i=1}^{m} \exp\{\sigma^k (A_i x^k - b_i)\}, \quad (4b)$$

$$c^T x^k - \sigma^k \sum_{i=1}^{m} \exp\{\sigma^k (A_i x^k - b_i)\} \geq c^T x^{k+1} - \sigma^k \sum_{i=1}^{m} \exp\{\sigma^k (A_i x^{k+1} - b_i)\}. \quad (4c)$$

Using inequalities (4a) and (4b), we obtain

$$c^T x^{k+1} - \sigma^{k+1} \sum_{i=1}^{m} \exp\{\sigma^{k+1}(A_i x^{k+1} - b_i)\} \geq c^T x^k - \sigma^k \sum_{i=1}^{m} \exp\{\sigma^k (A_i x^k - b_i)\}.$$

This means that

$$E(x^{k+1}, \sigma^{k+1}) \geq E(x^k, \sigma^k),$$

as required in the theorem. Hence, the theorem is established. ■

Using definition of $E(x^k, \sigma^k)$ and Theorem 3, we have

$$c^T x^{k+1} \geq E(x^{k+1}, \sigma^{k+1}) \geq E(x^k, \sigma^k). \quad (5)$$

It follows that

$$f^* \geq \ldots \geq E(x^{k+1}, \sigma^{k+1}) \geq E(x^k, \sigma^k). \quad (6)$$

Let $\bar{x}$ be a limit point of $\{x^k\}$. Since $x^k \geq 0$ and $\{x \in R^n | x \geq 0\}$ is a closed set we obtain that $\bar{x} \geq 0$. In another way, by using continuity of function $A_i x - b_i$ for all $i = 1,\ldots,m$, we see that

$$\sigma^k \sum_{i=1}^{m} \exp\{-\sigma^k (A_i x^k - b_i)\} \to 0 \text{ as } k \to \infty \quad (7)$$

If and only if

$$A_i x^k - b_i \to 0 \text{ for } i = 1,\ldots,m,$$

thus, $A\bar{x} = b$. Hence, $\bar{x}$ is feasible, and

$$f^* \geq c^T \bar{x}.$$

Using (6), the sequence of $\{E(x^k, \sigma^k)\}$ of exponential barrier function values is non-decreasing and bounded from above, and must converge monotonically from below to a limit, say $g^* := c^T \bar{x}$. Suppose that $g^* < f^*$. In this case, we define a positive number

$$\gamma = \tfrac{1}{2}(f^* - g^*).$$

It follows from $c^T x^k \to c^T \bar{x} = g^*$ that there exists a positive real number k_0 such that for all $k \geq k_0$,

$$c^T x^k < g^* - \gamma. \quad (8)$$

From (7), there exists k_1 such that for all $k \geq k_1$,

$$\sigma^k \sum_{i=1}^{m} \exp\{-\sigma^k (A_i x^k - b_i)\} < \tfrac{1}{2}\gamma. \quad (9)$$

If we apply (8)–(9) and take $k \geq \max\{k_0, k_1\}$, the result is

$$\begin{aligned} E(x^k, \sigma^k) &= c^T x^k - \sigma^k \sum_{i=1}^{m} \exp\{\sigma^k (A_i x^k - b_i)\} \\ &< g^* - \gamma + \tfrac{1}{2}\gamma \\ &= g^* - \tfrac{1}{2}\gamma. \end{aligned} \quad (10)$$

Taking $k \to \infty$ and using (10), we have

$$g^* < g^* - \tfrac{1}{2}\gamma,$$

that is,

$$\gamma < 0,$$

which contradicts with the assumption that $\gamma > 0$. We conclude that $g^* = f^*$ and $E(x^k, \sigma^k) \to f^*$ as $k \to \infty$, which gives result to the following theorem.

Theorem 4 (Convergence of exponential barrier function)
Let $\{\sigma^k\}$ be an increasing sequence of positive barrier parameters such that $\sigma^k \to \infty$ as $k \to \infty$. Let $\{x^k\}$ is a sequence of the maximizer of the problem (3) associated with σ^k. Then
(a) $c^T x^k \to f^*$ as $k \to \infty$.
(b) $E(x^k, \sigma^k) \to f^*$ as $k \to \infty$. ■

III. ALGORITHM

The implication of this theorem is remarkably strong. For any linear programming, the exponential barrier function has a finite unconstrained maximizer for every value of the barrier parameter, and every limit point of a maximizing sequence for the barrier function is a constrained maximizer of a problem (1). Based on the Theorem 4 we construct an algorithm for solving problem (1).

Algorithm 1
Given $Ax = b$, $\sigma^1 > 0$, the number of iteration N and $\varepsilon > 0$.

1. Choose $x^1 \in R^n$ such that $Ax^1 = b$ and $x^1 > 0$.
2. If the optimality conditions are satisfied for problem (1) at x^1, then stop.
3. Compute $E(x^1, \sigma^1) := \max_{x \geq 0} E(x, \sigma^1)$ and the maximizer x^1.
4. Compute $E(x^k, \sigma^k) := \max_{x \geq 0} E(x, \sigma^k)$, the maximizer x^k and $\sigma^k := 10\sigma^{k-1}$ for $k = 2$.
5. If $\| x^k - x^{k-1} \| < \varepsilon$ or $| E(x^k, \sigma^k) - E(x^{k-1}, \sigma^{k-1}) | < \varepsilon$ or $k = N$ then stop, else $k := k + 1$ and go to step 4. ■

IV. INTERIOR–POINT ALGORITHM

This section reviews the interior–point algorithm called Karmarkar's algorithm for finding a solution of linear programming problem. The step of this algorithm can be summarized as follows for any iteration (Parwadi, 2011).

Step 1. Given the current initial trial solution $(x_1, x_2, \ldots, x_n)$, set

$$D = \begin{pmatrix} x_1 & 0 & 0 & \ldots & 0 \\ 0 & x_2 & 0 & \ldots & 0 \\ 0 & 0 & x_3 & \ldots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & 0 & x_n \end{pmatrix}.$$

Step 2. Calculate $\tilde{A} = AD$ and $\tilde{c} = Dc$.

Step 3. Calculate $P = I - \tilde{A}^T (\tilde{A}\tilde{A}^T)^{-1} \tilde{A}$ and $c_p = P\tilde{c}$.

Step 4. Identify the negative component of c_p having the largest absolute value, and set ν equal to this absolute value. Then calculate

$$\tilde{x} = [1 \quad 1 \quad \ldots \quad 1]^T + \frac{\alpha}{\nu} c_p,$$

where α is a selected constant between 0 and 1.

Step 5. Calculate $x = D\tilde{x}$ as the trial solution for the next iteration (step 1). (If this trial solution is virtually unchanged from the preceding one, then the algorithm has virtually converged to an optimal solution, so stop.)

V. NUMERICAL EXAMPLES

This section we give five examples to test the Algoirthm 1 and we compare the results with Karmarkar's algorithm. Consider the following problems (Parwadi, 2010).

Example 1.
Maximize
$$f = 2x_1 + 5x_2 + 7x_3$$
subject to
$$x_1 + 2x_2 + 3x_3 = 6,$$
$$x_j \geq 0, \text{ for } j = 1, 2, 3.$$

Example 2.
Maximize
$$f = 0.4x_1 + 0.5x_2$$
subject to
$$0.3x_1 + 0.1x_2 \leq 2.7,$$
$$0.5x_1 + 0.5x_2 = 6,$$
$$x_j \geq 0,$$
for $j = 1, 2$.

Example 3.
Maximize
$f = -3x_1 + 4x_2$
subject to $x_1 - x_2 \le 0$,
$-x_1 + 2x_2 \le 2$,
$x_j \ge 0$, for j = 1, 2.

Example 4.
Maximize
$f = 4x_1 + 3x_2$
subject to $2x_1 + 3x_2 \le 6$,
$4x_1 + x_2 \le 4$,
$x_j \ge 0$, for j = 1, 2.

Tabel 1. Algorithm 1and Karmarkar's Algorithm Test Statistics

Problem Number	Algorithm 1		Karmarkar's Algorithm	
	Total Iterations	Time (Secs.)	Total Iterations	Time (Secs.)
1.	11	7.6	16	3.6
2.	8	5.9	19	3.7
3.	10	8.9	19	3.7
4.	12	10.9	12	2.8

Table 1 reports the results of computational for Algorithm 1 and Karmarkar's Algorithm. The first column of Table 1 contains the example number and the next two columns of each algorithm in this table contain the total iterations and the times (in seconds) of each algorithm. Table 1 also shows that in terms of the number of iterations required to complete the fourth numerical examples, the Algorithm 1 is better than Karmarkar's Algorithm, but in terms of completion time required to complete the four numerical examples shows that Karmarkar's algorithm looks better than Algorithm 1.

VI. CONCLUSION

From the discussion in previous section we see that this paper describes the exponential barrier function to solve problem (1). We also see that the maximum value of the exponential barrier function converges from above to the solution of problem (1) as the barrier parameter converges to infinity. Moreover, the maximizers of exponential barrier functions converge from the right to the maximizer of problem (1). The algorithm for this method ia also given in this paper. We also note the important thing of these methods which do not need an interior point assumption.

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