A Variation on Planarized QuasiPAS Model

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Bachtiar (2009) and Bachtiar and James (BJ, 2012) proposed a new kinematic dynamo model called by Quasipas model. Quasipas flow is a modification of Pekeris, Accad, and shkollar flow. They proposed this flow in their effort to find planar velocity dynamos. In this work, we revisited the fully planarized version of Quasipas model. Based on the results of Bachtiar (2009) and BJ (2012), we tried to increase the portion of planarized poloidal part of this flow. We suspected that planarized poloidal part plays an important role in improving the magnetic field's rate. We found that the magnetic field's rate of fully planarized Quasipas has improved, however we could not find any successful dynamo.

Keywords: Planar Velocity, Quasipas flow, planarized poloidal part

I. Introduction

It is generally believed that the Earth's magnetic field (EMF) is produced by a dynamo action taken place in the Earth's interior. This dynamo is a special type of dynamo, which called by self excited dynamo. This means that the dynamo can maintain the magnetic field independently. It was Larmor who proposed the Earth's self excited dynamo. It is known that the conductor fluid in the Earth's outer core is one of the Earth's dynamo components. From Paleomagnetism, we can observe that EMF has a reversal property. Considering those facts, Magnetohydrodynamics is considered as an appropriate model for the Earth's dynamo. Solving MHD is a very complicated project. We need to solve a system consists of six equations.

Instead of solving MHD, many scholars try to simplify the problem by solving one of the equations, which is the magnetic induction equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = R\nabla \times (\mathbf{u} \times \mathbf{B}) + \nabla^2\mathbf{B} \qquad (1)$$

Where:
B is the Magnetic field
u is the Velocity field
R (=UL/ η) is the magnetic Reynolds number

If we specified the velocity, the problem is called by kinematic dynamo. In this problem, we try to find flows that can produce dynamo process. The interaction between the given flow and initial magnetic field should dominate the magnetic diffusion term, which is the last term in equation (1).

After Lamor introduced the Earth's self excited dynamo, Childress reported that there are certain conditions that make the self excited dynamo cannot be produced. These conditions are called by antidynamo theorem. However, some scholars successfully found flows that can maintain the magnetic field. Pekeris, Accad, and Shkollar (PAS, 1973) is one of the earliest successful dynamo. The solution of PAS dynamo converges at relatively low magnetic Reynold number. It indicates that PAS flow can easily maintain the magnetic field.

One of the antidynamo theorems is Planar velocity antidynamo theorem (PVT). PVT precludes the existence of a dynamo with planar velocity conductor. Zel'dovich proposed PVT in 1957. However, he only proved the theorem when the conductor occupies all space. In 2006, Bachtiar, Ivers and James (BIJ 2006) showed that it is not impossible to prove PVT in finite volume. Moreover, BIJ investigate PVT, with finite volume, numerically. Surprisingly, one of their 32 models shows an early indication the existence of planar velocity dynamos.

II. Planar Velocity dynamos

In their investigation, BIJ also transformed several successful dynamos into planar velocity dynamo using their formula. One of them was PAS dynamo. BIJ found that PAS flow cannot be transformed into planar flow because it cannot satisfy the consistency condition:

$$\int_0^1 r^{n+2} t_n dr = 0 \qquad (2)$$

This means that the planarized PAS is not able to satisfy the differentiability and rigid boundary condition simultaneously.

Bachtiar (2009) and BJ (2012) modified PAS flow so that it can be planarized. They proposed two new flows: Quasipas and Bipas. Unfortunately, they did not find any successful dynamo using planarized Quasipas and Bipas. But, they found some new dynamo using the original and partly planarized of Quasipas and Bipas. Moreover, Bachtiar (2009) showed that both Quasipas and Bipas flow can produce dynamo action when the poloidal part is planarized. BJ (2012) claim that Quasipas and Bipas flow can also produce dynamo action when the toroidal part is planarized. However, it requires higher truncation level.

In this work, we try to reinvestigate planarized Quasipas model. We argue that the planarized poloidal part plays an important role in improving the magnetic field.

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In our simulation, we increase the portion of planarized poloidal part and observe the change in the magnetic field.

III. MATHEMATICAL BACKGROUND

In solving the induction equation, we expand the magnetic and velocity field into toroildal and poloidal form and use spherical harmonics expansion. This method was introduced by Bullard and Gellman in 1954. We assume that the magnetic field is in the form $\mathbf{B}= \sum_\lambda \mathbf{B}_\lambda e^{\lambda t}$ and the velocity field does not depend in time.

$$\mathbf{B} = \sum_{n,m}(\mathbf{T}_n^m + \mathbf{S}_n^m)$$

$$\mathbf{u} = \sum_{n,m}(\mathbf{t}_n^m + \mathbf{s}_n^m)$$

where

$$\mathbf{T}_n^m = \nabla\times(rT_n^m(r,t)Y_n^m(\theta,\varphi))$$

$$\mathbf{S}_n^m = \nabla\times\nabla\times(rS_n^m(r,t)Y_n^m(\theta,\varphi))$$

$$n = 1,2,3,\ldots; m = -n,\ldots,n.$$

$$Y_n^m = (-)^m\left[\frac{2n+1}{2-\delta_m^0}\right]^{\frac{1}{2}} P_n^m(\cos\theta)e^{im\varphi}$$

$$= (-)^m\overline{Y_n^{-m}}$$

$$Y_n^m = 0 \quad , for \quad n < |m|$$

To discretise the radial direction, we use central difference scheme. With a certain arrangement, we can simplify the problem into an ordinary eigenvalue problem. Our goal is to get a positive real part of the eigenvalue, for each value of magnetic Reynolds number, which indicates the growing mode of **B** and the dynamo process exists. The detail of the mathematical background can be seen in many references such as Dudley and James (1989) and BIJ (2006). The discussion of the spherical harmonic expansion can be seen in James (1974).

IV. QUASIPAS

Bachtiar (2009) and BJ (2012) define QuasiPAS flow, as the following:

$$\mathbf{u} = 2\,Re\,\{\mathbf{s}_2^2 + \mathbf{t}_2^2\}$$

Where

$$\mathbf{s}_2^2(r) = \nabla\times\nabla\times(s_2^2Y_2^2)$$
$$\mathbf{t}_2^2(r) = \nabla\times(t_2^2Y_2^2)$$
$$s_2^2 = K\Lambda_i j_2(\Lambda_i r)$$
$$t_2^2 = K\Lambda_i^2 j_2(\Gamma_i r)$$

$\Lambda = 5.3674, 9.0950, 12.3229,$
$\Gamma = 6.98793, 10.4171, 13.6980.$

This flow satisfies the rigid boundary condition, slip condition, differentiability condition. In addition, this flow also satisfies the consistency condition (2) so that it can be fully planarized using BIJ's formula. ***Λ*** and ***Γ*** are the first three positive root of spherical Bessel function order two and three.

If we planarize Quasipas using BIJ's formula, we will get the following flow

$$\mathbf{u} = 2\,\mathrm{Re}\left\{\varepsilon_s\left(\mathbf{s}_2^2+\mathbf{t}_3^2\right)+\varepsilon_t\left(\mathbf{t}_2^2+\mathbf{s}_3^2+\mathbf{t}_4^2\right)\right\},$$

where

$$\mathbf{s}_2^2(r) = \nabla\times\nabla\times(s_2^2Y_2^2)$$
$$\mathbf{t}_3^2(r) = \nabla\times(t_3^2Y_3^2)$$
$$\mathbf{t}_2^2(r) = \nabla\times(t_2^2Y_2^2)$$
$$\mathbf{s}_3^2(r) = \nabla\times\nabla\times(s_3^2Y_3^2)$$
$$\mathbf{t}_4^2(r) = \nabla\times(t_4^2Y_4^2)$$

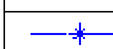
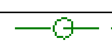
$$s_2^2(r) = K\Lambda_i j_2(\Lambda_i r)$$
$$t_2^2(r) = K\Lambda_i^2 j_2(\Gamma_k r)$$
$$t_3^2(r) = -\alpha_3 K\Lambda_i j_3(\Lambda_i r)$$
$$s_3^2(r) = \frac{i}{2\alpha_3}\frac{\Lambda^2{}_i}{\Gamma_k}K j_3(\Gamma_k r))$$
$$t_4^2(r) = 0.5\sqrt{3}K\Lambda^2{}_i j_4(\Gamma_k r),$$

$\mathbf{s}_2^2+\mathbf{t}_3^2$ is the planarized poloidal part and $\mathbf{t}_2^2+\mathbf{s}_3^2+\mathbf{t}_4^2$ is the planarized toroidal part. For the original planarized Quasipas $\varepsilon_s = \varepsilon_t = 1$. However, we will use various values of ε_s and ε_t in order to achieve our goals. We will discuss in more detail in the next section.

V. DISCUSSION

Instead of increasing the planarized poloidal part, i.e $\varepsilon_s > 1$ and $\varepsilon_t = 1$, we decided to decrease the planarized toroidal part, i.e. $\varepsilon_s = 1$ and $\varepsilon_t < 1$. Increasing the planarized poloidal part means that the flow becomes greater in magnitude. As a result, we may need smaller grid in all direction. We do want to avoid this problem due to our limited computational resource. On the other hand, decrasing the planarized toroidal part will make the flow becomes smaller in magnitude. Consequently, we expect the model will converge easier that the original planarized Quasipas.

In this work, we used $\varepsilon_t = 0.1, 0.5, 0.9$, $0 \le R \le 0.45$ $[J,N] = [100,11]$and four models of planarized Quasipas with $\Lambda_i\,\Gamma_j = \Lambda_1\,\Gamma_2,\ \Lambda_1\,\Gamma_3,\ \Lambda_2\,\Gamma_2, \Lambda_2\,\Gamma_3.$ The following are profiles for dominant eigenvalue :

	Original planarized Quasipas
	Planarized Quasipas with increment on planarized poloidal part

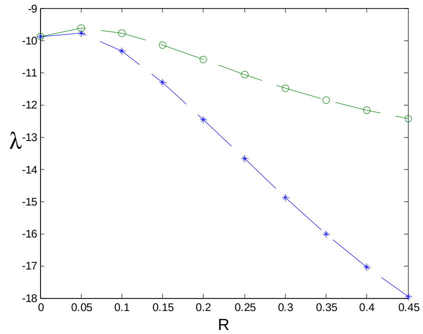
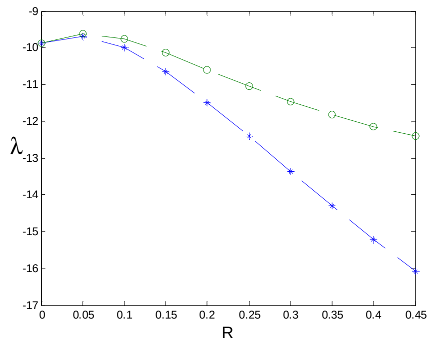
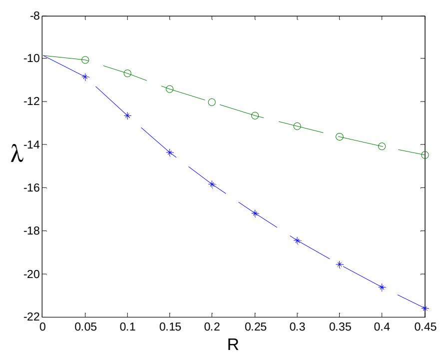
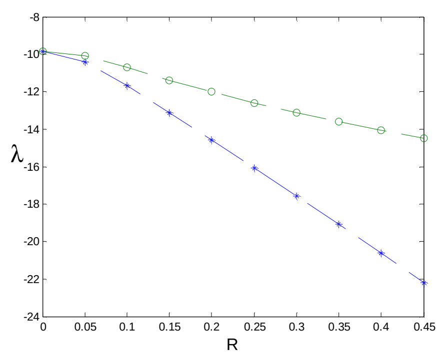
$\Lambda_1\,\Gamma_2$ $\varepsilon_t = 0.1, \varepsilon_s = 1$	
$\Lambda_1\,\Gamma_3$ $\varepsilon_t = 0.1, \varepsilon_s = 1$	
$\Lambda_2\,\Gamma_2$ $\varepsilon_t = 0.1, \varepsilon_s = 1$	
$\Lambda_2\Gamma_3$ $\varepsilon_t = 0.1, \varepsilon_s = 1$	

Table 1. Plots of dominant eigenvalue of Quasipas with 90% increment on planarized poloidal part.

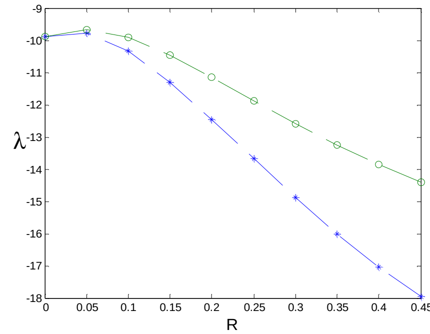
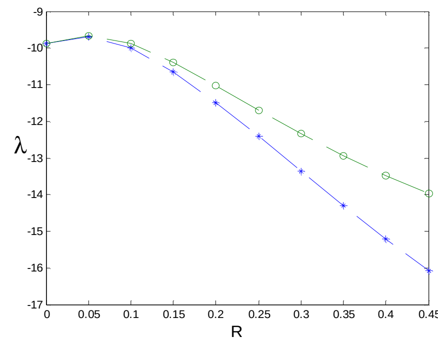
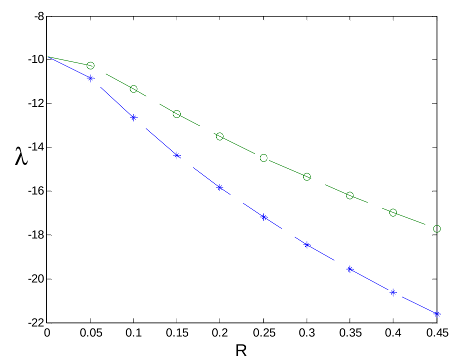
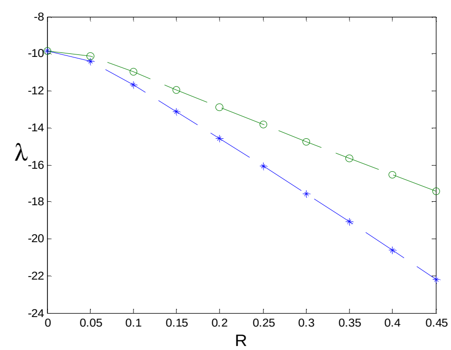
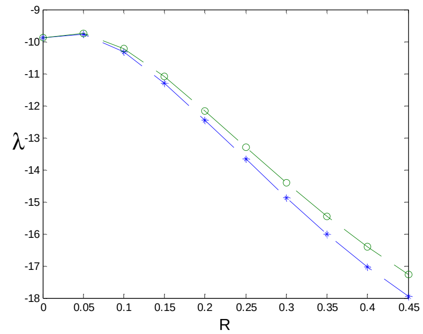
$\Lambda_1\,\Gamma_2$ $\varepsilon_t = 0.5, \varepsilon_s = 1$	
$\Lambda_1\,\Gamma_3$ $\varepsilon_t = 0.5, \varepsilon_s = 1$	
$\Lambda_2\,\Gamma_2$ $\varepsilon_t = 0.5, \varepsilon_s = 1$	
$\Lambda_2\Gamma_3$ $\varepsilon_t = 0.5, \varepsilon_s = 1$	

Table 2. Plots of dominant eigenvalue of Quasipas with 50% increment on planarized poloidal part.

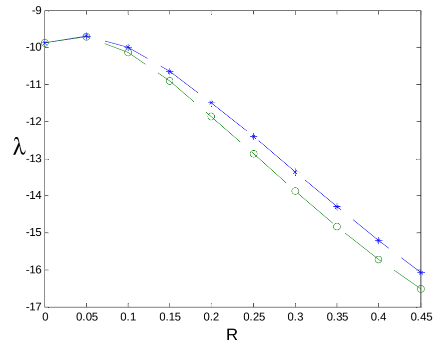
$\Lambda_1\,\Gamma_2$

$\varepsilon_t = 0.9,\ \varepsilon_s = 1$



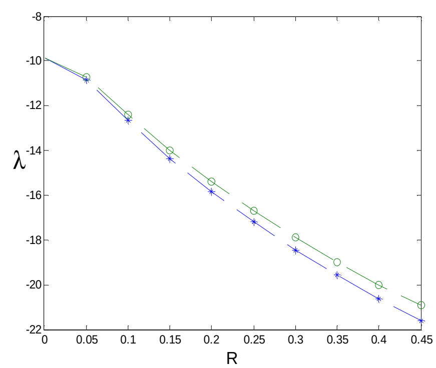
$\Lambda_1\,\Gamma_3$

$\varepsilon_t = 0.9,\ \varepsilon_s = 1$



$\Lambda_2\,\Gamma_2$

$\varepsilon_t = 0.9,\ \varepsilon_s = 1$



$\Lambda_2\Gamma_3$

$\varepsilon_t = 0.9,\ \varepsilon_s = 1$

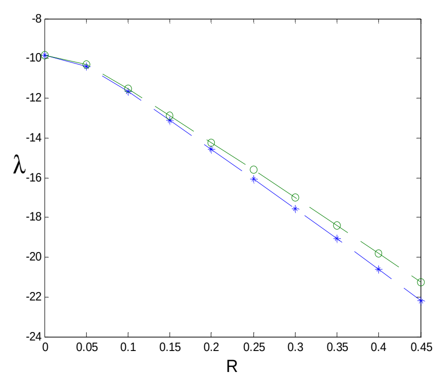


Table 3. Plots of dominant eigenvalue of Quasipas with 50% increment on planarized poloidal part.

Model	$\varepsilon_t = 0.1, \varepsilon_s = 1$	$\varepsilon_t = 0.5, \varepsilon_s = 1$	$\varepsilon_t = 0.9, \varepsilon_s = 1$
$\Lambda_1\,\Gamma_2$	9.7523	6.4254	1.2634
$\Lambda_1\,\Gamma_3$	5.9696	3.3318	1.2167
$\Lambda_2\,\Gamma_2$	14.3367	8.2342	1.4816
$\Lambda_2\Gamma_3$	13.2810	8.3206	1.6413

Table 4. Maximum difference between the dominant eigenvalue of original planarized Quasipas and planarized Quasipas with increment on planarized poloidal part.

Tables 1-3 show that increasing the portion of planarized poloidal part improves the magnetic field's profile. If we impose larger increment on planarized poloidal part, then we got better profile. The maximum difference of the dominant eigenvalue can be seen in Table 4. In all cases, 90% increment have maximum difference in the dominant eigenvalue. However, within the Reynolds number interval that we observed, there is no indication of successful dynamo.

VI. Conclusion

Our results show that the planarized poloidal part plays an important role in improving the magnetic field's profiles. Although, we are not able to find any successful dynamos, our results will be important for us to determine our next project.

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