On the Catastrophic model and Stability

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Abstract
In this paper, we have applied the method of averaging to obtain periodic solutions of some nonlinear differential equation (NLDE). Then we have determined the averaging system to obtain the stability of periodic solution of NLDE. And the main result is the following proposition:
The butterfly type catastrophe occurs in case of NLDE of fifth degree.

Keywords: Cusp catastrophe model, cusp type catastrophe, nonlinear differential equations, saddle-node bifurcation..

1 Introduction

Most phenomena in our world are essentially nonlinear and are described by nonlinear ordinary differential equations. Solving nonlinear ordinary differential equation is thus of great importance for gaining insight into real-world or engineering problems. However, generally speaking, it is difficult to obtain accurate solutions of nonlinear problems[1].So I attempt to solve some nonlinear differential equations by combining catastrophe theory and Krylov-Boogoliubov method.
Catastrophe theory is the study of singularity, discontinuity and it is about the possible shapes of the equilibrium states form a catastrophic manifold when we have n state variables in the vector x and k parameters so the possible equilibrium states form this surface, or catastrophic manifold in (n + k)-dimensional space[2]. The behavior of the system over time is described by trajectories over this manifold, so it is important to us to find such a manifold. As well known there are seven types of elementary catastrophe:

Fold, Cusp, swallowtail, Hyperbolic umbilici, Elliptic, Butterfly , Parabolic umbilici[2]. We have studded here the butterfly catastrophe and we have related this type with NLDE involves the study of change using the techniques of catastrophe theory[3] spatially we have used the method of Krylov and Bogoliubov to study the stability of periodic solution by using new conditions for the second order nonlinear differential equations:

$$\ddot{x}+\omega^2 x+\varepsilon f\,(x.\dot{x})=0$$

We have shown here that the butterfly catastrophe occurs when the NLDE of fifth degree,. We have divided the main body of this work into three parts; the first part is the introductory in section 2 we have described the method of Krylov and Bogoliupov. in the section 3 we have studded the stability of limit cycles and in section 4 we have studded catastrophic manifold of butterfly.

2. The Method of Krylov and Bogoliubov

Although Krylov and Bogoliubov’s method is fairly general, we will apply it only to equations of the form:

$$\ddot{x}+\omega^2 x+\varepsilon f\,(x.\dot{x})=0 \qquad (2.1)$$

where ε is small parameter.

For the case ε = 0 we may apply linear theory to obtain the solution:

$$x = A\sin(\omega t + \phi)$$

where A and Ø are arbitrary constants. Differentiating gives:

$$\dot{x} = A\omega\cos(\omega t + \phi)$$

Assume that, for small ε, the solution of (2.1) takes the form:

$$x = A(t)\sin(\omega t + \phi(t))$$

$$\dot{x} = A(t)\omega\cos(\omega t + \phi(t))$$

where A(t) and Ø(t) are slowly varying functions of t.

We proceed to obtain the approximate solution of eq. (1) as follows:

Let $y = \dot{x}$ (2.2)

and, from eqs. (2.1) and (2.2), we have

$$\dot{y} = -\omega^2 x - \varepsilon f(x, y) \qquad (2.3)$$

To satisfy eqs. (2.2) and (2.3), we further assume that

$$x = A(t)\sin(\omega t + \phi(t))$$

$$\dot{x} = A(t)\omega\cos(\omega t + \phi(t)) \qquad (2.4)$$

where $A(t)$ *and* $\phi(t)$ are slowly varying functions of t, and therefore $\overset{**}{A}$ *and* $\overset{**}{\phi}$ can be neglected. In order that the set of equations (2.4) should be the solutions of equations (2.2) and (2.3) it must satisfy the following conditions

$$\overset{*}{A}\sin\Psi + A\overset{*}{\phi}\cos\Psi = 0 \qquad (2.5)$$

and .

$$\dot{A}\omega\cos\Psi - A\omega(\omega + \dot{\phi})\sin\Psi = -\omega^2 A\sin\Psi - \varepsilon f(x, y).$$

Therefore

$$\dot{A}\cos\Psi - A\dot{\phi}\sin\Psi = -\frac{\varepsilon}{\omega} f(A\sin\Psi, A\omega\cos\Psi) \qquad (2.6)$$

Where $\Psi = \omega t + \phi$

Solving (2.5) and (2.6) for $\dot{A}$ *and* $\dot{\phi}$, we get:

$$\dot{A} = -\frac{\varepsilon}{\omega}\cos\Psi f(A\sin\Psi, A\omega\cos\Psi) \qquad (2.8)$$

$$\dot{\phi} = \frac{\varepsilon}{\omega A}\sin\Psi f(A\sin\Psi, A\omega\cos\Psi) \qquad (2.9)$$.

Note that A and Ø are both proportional to ε, conferencing that A and Ø are slowly varying functions of time when ε is small and that in terms of the assumption contained in (2.2) and (2.3) equations (2.8)and (2.9) are exact representation of A and Ø.

Krylov and Bogoliuboves approximation is to replace A and Ø in equations 2.8 and 2.9 by their average values over one period 2π/ω. A is regarded as a constant in taking the average.

This procedure (known as a method of averaging) leads to

$$\dot{A} = -\frac{\varepsilon}{2\pi}\int_0^{2\pi/w}\cos\Psi f(A\sin\Psi, A\omega\cos\Psi)dt \qquad (2.10)$$

$$\dot{\phi} = \frac{\varepsilon}{2\pi A}\int_0^{2\pi/w} \sin\Psi f(A\sin\Psi, A\omega\cos\Psi)dt \qquad (2.11)$$

Because $d\Psi = \omega dt$, the substitution $\Psi = \omega t + \phi$ gives the final results

$$\dot{A} = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} \cos\Psi\, f(A\sin\Psi, A\omega\cos\Psi)d\Psi \qquad (2.12)$$

$$\dot{\phi} = \frac{\varepsilon}{2\pi A\omega}\int_0^{2\pi} \sin\Psi\, f(A\sin\Psi, A\omega\cos\Psi)d\Psi \qquad (2.13)$$

The exact equations 2.8 and 2.9 are thus replaced by approximate equations 2.10 and 2.11. Once the integrals have been evaluated we have first order differential equations to solve for A and Ø. We should find the values of A and Ø by evaluating the integrals 2.12 and 2.13. Then the solution is given approximately by x=A sin (ωt+Ø) whenever A and Ø take their values.

3. Stability of Limit cycles

The amplitudes of possible limit cycles are given by solutions of the equation

$$\dot{A} = 0, \ \ i.e.\ \ A\ is\ \text{constant}$$

Now

$$\dot{A} = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} \cos\Psi\ \ f(A\sin\Psi, A\omega\cos\Psi)d\Psi = G(A),\ say$$

So the amplitudes of the limit cycles are given by the solutions of G(A)=0.
The equation arises whether or not the limit cycle is stable, i.e. if we made a slight disturbance from the limit cycle trajectory in the phase – plane, would the motion return to diverge from the limit cycle.

Consider the expression for A. Suppose that a solution of G(A) = 0 is $A = A_1$.
A_1 is the amplitude of a limit cycle, and $G(A_1) = 0$. Now make the disturbance $A = A_1 + \eta$ where η is small. For a stable limit cycle, we require $\eta \to 0$ as $t \to \infty$.

Differentiating we obtain $\dot{A} = \dot{\eta}$.

Also $\dot{A} = G(A_1 + \eta)$

$$\approx G(A_1) + \eta G'(A_1)$$

$$= \eta\ G'(A_1), \text{ since } G(A_1) = 0.$$

So $\dot{\eta} \approx \eta G'(A_1)$. Solving this equation gives
$\eta \approx Ce^{G(A_1)t}$,where C is an arbitrary constant.
So $\eta \to 0$ as $t \to \infty$ provided $G(A_1) < 0$.
We now have a condition for stability:

1- If $G'(A_1) < 0$, there is a stable limit cycle at $A = A_1$.
2- If $G'(A_1) > 0$, there is an unstable limit cycle at $A = A_1$.

4. Catastrophic manifold of butterfly

Our purpose, in this section, is to find the catastrophic manifold of butterfly catastrophe, and to show that the butterfly catastrophe occurs in case of NLDE of fifth degree. To do this we first define the function f that represents the butterfly catastrophe. Suppose that the possible equilibrium states of the system are the minima of the function f(x) given by

$$f(x) = x^6 + u_1x^4 + u_2x^3+u_3x^2+u_4x \quad (4.1)$$

The stationary values are given by

$$\partial f/\partial x = 6x^5+4u_1x^3+3u_2x^2+2u_3x+u_4 = 0 \quad (4.2)$$

The equation (4.2) can have one or three or five real roots.

The second derivative $\partial^2 f/\partial x^2$ is zero on some curve (which includes the point (0, 0, 0, 0, 0) in (x, u_1, u_2, u_3, u_4) – space), and hence that the function has degenerate singularities along the curve. Also the second derivative can be used to identify the minima; in the case of three real roots, two are minima; and in the case of the single real root, that turns out to be a minimum.

We consider the non-linear differential equation (which is of fifth degree)

$$\ddot{x} + \omega^2 x = \frac{6}{5x^5} \quad 4.3$$

The averaged system is

$$\dot{A} = 5/16\ \varepsilon\omega^4 A^5+1/2\varepsilon A^3+\beta^2 A \quad (4.4)$$

The catastrophic manifold for the averaged system (4.4) is

$5/16\ \varepsilon\omega^4 A^5+1/2\varepsilon A^3+\beta^2 A = 0$

which represents a butterfly catastrophic model . Hence the catastrophic phenomena appears in this system is butterfly.

Then the following proposition holds:

The butterfly type catastrophe occurs in case of NLDE of fifth degree.

References

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