TWO-DIMENSIONAL FRACTIONAL SYSTEM OF NONLINEAR DIFFERENCE EQUATIONS IN THE MODELING COMPETITIVE POPULATIONS

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Abstract

In this paper we have already investigated the solutions of the two-dimensional fractional system of nonlinear difference equations in the modeling competitive populations in the form

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1}y_n + \alpha} \quad \& \quad y_{n+1} = \frac{y_{n-1}}{y_{n-1}x_n + \beta} \qquad (1)$$

where α and β are real numbers with the initial conditions x_{-1}, x_0, y_{-1} and y_0 such that $x_{-1}y_0 \neq \alpha$ and $y_{-1}x_0 \neq \beta$. Moreover, we have studied the local stability, global stability, boundedness and periodicity of solutions. We will consider some special cases of (1) as applications. Finally, we give some numerical examples.

Keywords*:* difference equation, solutions ,convergence ,periodicity ,eventually periodic, competitive, high orders, stability.

Mathematics Subject Classification*:* ***39A10****,* ***39A11***

1 Introduction

Difference equations or discrete dynamical systems [38] is diverse field which impact almost every branch of pure and applied mathematics. Every dynamical system $a_{n+1} = f(a_n)$ determines a difference equation and vise versa .Recently, there has been great interest in studying difference equations systems. One of the reasons for this is a necessity for some techniques whose can be used in investigating equations arising in mathematical models [21] describing real life situations in population biology [17], economic, probability theory, genetics , psychology, ...etc .

The study of properties of rational difference equations [24] and systems of rational difference equations has been an area of interest in recent years, see book [20] and the references therein (see also [1] , [15]).

A first order system of difference equations

$$x_{n+1} = f(x_n, y_n)$$
$$y_{n+1} = g(x_n, y_n) \quad (2)$$

where , n $=0,1, \ldots \ldots$, $(x_0, y_0) \in R$, R subset of the real plane , $(f, g): R \to R$, f, g are continuous function is *competitive* if $f(x, y)$ is non-decreasing in x and non-increasing in y ; and $g(x, y)$ is non-increasing in x and non-decreasing in y . System (2) where the functions f and g have monotonic character opposite of the monotonic character in competitive system will be called *anti-competitive* . It is well know that the dynamical properties of competitive populations has received great attention from both theoretical and mathematical biologists [39] due to its universal prevalence and important. Competitive and anti-competitive systems were studied by many authors (see for examples [3], [4], [11],[12], [17],[18], [22],
[27], [28], [29], [39] , [40], [41]).

In a modeling setting, the two-dimensional competitive system of nonlinear rational difference equations

$$x_{n+1} = \frac{x_n}{a + y_n} \quad \& \quad y_{n+1} = \frac{y_n}{b + x_n}$$

represents the rule by which two discrete, competitive populations reproduce from one generation to the next. The phase variables x_n and y_n denote population sizes during the n-th generation and sequence or orbit $\{(x_n, y_n): n = 0, 1, \ldots\}$ describes how the populations evolve over time. Competitive between the populations is reflected by the fact the transition function for each population is a decreasing function of the other population
size. For instance , In [22] M.P. Hassell, H.N. Comins studied a discrete (difference) single age-class model for two-species competition and its stability properties discussed .

There are many papers in which systems of difference equations have studied .
Cinar et al. [5] has obtained the positive solution of the difference equation system

$$x_{n+1} = \frac{m}{y_n} \quad \& \quad y_{n+1} = \frac{p y_n}{x_{n-1} \, y_{n-1}}$$

Cinar [6] has obtained the positive solution of the difference equation system

$$x_{n+1} = \frac{1}{y_n} \quad \& \quad y_{n+1} = \frac{y_n}{x_{n-1} \, y_{n-1}}$$

Also, Cinar [7] has obtained the positive solution of the difference equation system

$$x_{n+1} = \frac{1}{z_n} \quad \& \quad y_{n+1} = \frac{x_n}{x_{n-1}} \quad \& \; z_{n+1} = \frac{1}{x_{n-1}}$$

Cinar [8]-[10] has got the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}$$

$$x_{n+1} = \frac{x_{n-1}}{-1 + x_n x_{n-1}}$$

$$x_{n+1} = \frac{a x_{n-1}}{1 + b \, x_n x_{n-1}}$$

Aloqeili [2] obtained the form of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}$$

Özban [35] has investigated the positive solutions of the system of rational difference equations

$$x_{n+1} = \frac{1}{y_{n-k}} \quad \& \quad y_{n+1} = \frac{y_n}{x_{n-m}\, y_{n-m+k}}$$

In [31] , Kurbanli studied a three-dimensional system of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1} \quad \& \quad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1} \quad \& \quad z_{n+1} = \frac{z_{n-1}}{y_n z_{n-1} - 1}$$

where the initial conditions are arbitrary real numbers. He expressed the solution of this system and investigated the behavior and computed for some initial values.

Elabbasy et al. [14] has obtained the solution for some particular cases of the following general system of difference equations

$$x_{n+1} = \frac{a_1 + a_2 y_n}{a_3 z_n + a_4 x_{n-1} z_n} \quad \& \quad y_{n+1} = \frac{b_1 z_{n-1} + b_2 z_n}{b_3 x_n y_n + b_4 x_n y_{n-1}} \quad \& \quad z_{n+1} = \frac{c_1 z_{n-1} + c_2 z_n}{c_3 x_{n-1} y_{n-1} + c_4 x_{n-1} y_n + c_5 x_n y_n}$$

Elsayed [16] investigated the solutions of the system of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{\pm 1 \pm x_{n-1} y_n} \quad \& \quad y_{n+1} = \frac{y_{n-1}}{\pm 1 + y_{n-1} x_n}$$

Although difference equations are sometimes very simple in their forms ,they are extremely difficult to understand thoroughly the behavior of their solutions . In book [26] V.L. Kocic, G. Ladas have studied global behavior of nonlinear difference equations of higher order . Similar nonlinear systems of rational difference equations were investigated (see [4],[39]). For some other recent papers on systems of difference equations, see, for examples, ([13] ,[24],[16],[19],[25],[30],[32],[33],[34],[36],[37],[43] ,[44]) and the related references therein.

Our goal , in this paper is to investigate the solutions of the two-dimensional fractional system of nonlinear difference equations in the modeling competitive populations in the form

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1} y_n + \alpha} \quad \& \quad y_{n+1} = \frac{y_{n-1}}{y_{n-1} x_n + \beta}$$

where α and β are real numbers with the initial conditions x_{-1}, x_0, y_{-1} and y_0 such that $x_{-1} y_0 \neq \alpha$ and $y_{-1} x_0 \neq \beta$. Moreover, we have studied the local stability, global stability, boundedness and periodicity of solutions. We will consider some special cases of (1) as applications. Finally, we give some numerical examples.

2 Solutions for System of Nonlinear Difference Equations in (1) :

The following theorem give the solution of the system of difference equation in (1)

Theorem 2.1: *Suppose that (x_n, y_n) be a solution of equation (1) where α and β are real numbers with the initial conditions x_{-1}, x_0, y_{-1}, and y_0 such that $x_{-1} y_0 \neq \alpha$ and $y_{-1} x_0 \neq \beta$. Then the solutions of equation (1) have the form :*

$$x_1 = \frac{x_{-1}}{A+\alpha}, y_1 = \frac{y_{-1}}{B+\beta}, x_2 = \frac{x_0 (B+\beta)}{B(1+\alpha)+\alpha\beta}, y_2 = \frac{y_0 (A+\alpha)}{A(1+\beta)+\alpha\beta} \quad (3)$$

$$x_{2n-1} = \frac{x_{-1}}{A+\alpha} \prod_{i=1}^{n-1} \left[\frac{A(1+\beta)\left\{ \sum_{j=0}^{i-1} (\alpha\beta)^j \right\} + (\alpha\beta)^i}{A(1+\alpha)\left\{ \sum_{j=0}^{i} (\alpha\beta)^j \right\} + (1-A)\left(\alpha^{i+1}\beta^i\right)} \right] \quad (4)$$

$$x_{2n} = \frac{x_0 (B+\beta)}{B+\alpha B+\alpha\beta} \prod_{i=1}^{n-1} \left[\frac{B(1+\beta)\left\{ \sum_{j=0}^{i} (\alpha\beta)^j \right\} + (1-B)\left(\alpha^i \beta^{i+1}\right)}{B(1+\alpha)\left\{ \sum_{j=0}^{i} (\alpha\beta)^j \right\} + (\alpha\beta)^{i+1}} \right] \quad (5)$$

$$y_{2n-1} = \frac{y_{-1}}{B+\beta} \prod_{i=1}^{n-1} \left[\frac{B(1+\alpha)\left\{ \sum_{j=0}^{i-1} (\alpha\beta)^j \right\} + (\alpha\beta)^i}{B(1+\beta)\left\{ \sum_{j=0}^{i} (\alpha\beta)^j \right\} + (1-B)\left(\beta^{i+1}\alpha^i\right)} \right] \quad (6)$$

$$y_{2n} = \frac{y_0 (A+\alpha)}{A+\beta A+\alpha\beta} \prod_{i=1}^{n-1} \left[\frac{A(1+\alpha)\left\{ \sum_{j=0}^{i} (\alpha\beta)^j \right\} + (1-A)\left(\beta^i \alpha^{i+1}\right)}{A(1+\beta)\left\{ \sum_{j=0}^{i} (\alpha\beta)^j \right\} + (\alpha\beta)^{i+1}} \right] \quad (7)$$

where $A = x_{-1}y_0$, $B = y_{-1}x_0$ *such that* $A \neq -\alpha$, $A \neq \frac{-\alpha\beta}{1+\beta}$, $B \neq -\beta$, $B \neq \frac{-\alpha\beta}{1+\alpha}$, $B \neq \frac{-\alpha\beta}{1+\alpha}$

and $n \geq 2$.

Proof

It is easy , from equations (1) , to see that (3) satisfies .

We will use the mathematical induction to prove the equations(4-7) .

By using equations(1) , for $n = 2$, we have

$$x_3 = \frac{x_1}{x_1 y_2 + \alpha} = \frac{\frac{x_{-1}}{A+\alpha}}{\left(\frac{x_{-1}}{A+\alpha}\right)\left(\frac{y_0 (A+\alpha)}{A(1+\beta)+\alpha\beta}\right)+\alpha} = \frac{\frac{x_{-1}}{A+\alpha}}{\left(\frac{x_{-1}y_0}{A(1+\beta)+\alpha\beta}\right)+\alpha}$$

$$= \frac{x_{-1}\left(A(1+\beta)+\alpha\beta\right)}{(A+\alpha)\left[A+\alpha\left(A(1+\beta)+\alpha\beta\right)\right]} = \frac{x_{-1}\left(A(1+\beta)+\alpha\beta\right)}{(A+\alpha)\left[A(1+\alpha+\alpha\beta)+\alpha^2\beta\right]} \quad (8)$$

Also

$$y_3 = \frac{y_1}{y_1 x_2 + \beta} = \frac{\frac{y_{-1}}{B+\beta}}{\left(\frac{y_{-1}}{B+\beta}\right)\left(\frac{x_0(B+\beta)}{B(1+\alpha)+\alpha\beta}\right)+\beta} = \frac{\frac{y_{-1}}{B+\beta}}{\left(\frac{B}{B(1+\alpha)+\alpha\beta}\right)+\beta}$$

$$= \frac{y_{-1}\{B(1+\alpha)+\alpha\beta\}}{(B+\beta)\left[B+\beta\{B(1+\alpha)+\alpha\beta\}\right]} = \frac{y_{-1}\{B(1+\alpha)+\alpha\beta\}}{(B+\beta)\left[B(1+\beta+\alpha\beta)+\alpha\beta^2\right]} \qquad (9)$$

By using (4) and (6) , for n = 2 , we have

$$x_3 = \frac{x_{-1}}{A+\alpha}\left[\frac{A(1+\beta)+\alpha\beta}{A(1+\alpha)\{1+\alpha\beta\}+(1-A)\alpha^2\beta}\right] = \frac{x_{-1}}{A+\alpha}\left[\frac{A(1+\beta)+\alpha\beta}{A(1+\alpha+\alpha\beta)+\alpha^2\beta}\right] \qquad (10)$$

Also

$$y_3 = \frac{y_{-1}}{B+\beta}\left[\frac{B(1+\alpha)+\alpha\beta}{B(1+\beta)\{1+\alpha\beta\}+(1-B)\beta^2\alpha}\right] = \frac{y_{-1}}{B+\beta}\left[\frac{B(1+\alpha)+\alpha\beta}{B(1+\beta+\alpha\beta)+\beta^2\alpha}\right] \qquad (11)$$

From equations (8)-(11) , the equations (4) and (6) hold at n = 2 .

Now by using equations(1) , for n = 3 , we have

$$x_4 = \frac{x_2}{x_2 y_3 + \alpha} = \frac{\frac{x_0(B+\beta)}{B(1+\alpha)+\alpha\beta}}{\left(\frac{x_0(B+\beta)}{B(1+\alpha)+\alpha\beta}\right)\left(\frac{y_{-1}\{B(1+\alpha)+\alpha\beta\}}{(B+\beta)\left[B(1+\beta+\alpha\beta)+\alpha\beta^2\right]}\right)+\alpha}$$

$$= \frac{x_0(B+\beta)\left[B(1+\beta+\alpha\beta)+\alpha\beta^2\right]}{\{B(1+\alpha)+\alpha\beta\}\left[B+\alpha\left[B(1+\beta+\alpha\beta)+\alpha\beta^2\right]\right]}$$

$$= \frac{x_0(B+\beta)\left[B(1+\beta+\alpha\beta)+\alpha\beta^2\right]}{\{B(1+\alpha)+\alpha\beta\}\left[B\left(1+\alpha+\alpha\beta+\alpha^2\beta\right)+\alpha^2\beta^2\right]} \qquad (12)$$

Also

$$y_4 = \frac{y_2}{y_2 x_3 + \beta} = \frac{\frac{y_0(A+\alpha)}{A(1+\beta)+\alpha\beta}}{\left(\frac{y_0(A+\alpha)}{A(1+\beta)+\alpha\beta}\right)\left(\frac{x_{-1}\{A(1+\beta)+\alpha\beta\}}{(A+\alpha)\left[A(1+\alpha+\alpha\beta)+\beta\alpha^2\right]}\right)+\beta}$$

$$= \frac{y_0(A+\alpha)\left[A(1+\alpha+\alpha\beta)+\beta\alpha^2\right]}{\{A(1+\beta)+\alpha\beta\}\left[A+\beta\left[A(1+\alpha+\alpha\beta)+\beta\alpha^2\right]\right]}$$

$$= \frac{y_0(A+\alpha)\left[A(1+\alpha+\alpha\beta)+\beta\alpha^2\right]}{\{A(1+\beta)+\alpha\beta\}\left[A\left(1+\beta+\alpha\beta+\beta^2\alpha\right)+\alpha^2\beta^2\right]} \qquad (13)$$

By using (5) and (7) , for n = 2 , we have

$$x_4 = \frac{x_0(B+\beta)}{B+\alpha B+\alpha\beta}\left[\frac{B(1+\beta)(1+\alpha\beta)+(1-B)\alpha\beta^2}{B(1+\alpha)(1+\alpha\beta)+(\alpha\beta)^2}\right]$$

$$= \frac{x_0(B+\beta)}{B+\alpha B+\alpha\beta}\left[\frac{B(1+\beta+\alpha\beta)+\alpha\beta^2}{B(1+\alpha+\alpha\beta+\alpha^2\beta)+(\alpha\beta)^2}\right] \tag{14}$$

Also

$$y_4 = \frac{y_0(A+\alpha)}{A+\beta A+\alpha\beta}\left[\frac{A(1+\alpha)(1+\alpha\beta)+(1-A)\alpha^2\beta}{A(1+\beta)(1+\alpha\beta)+(\alpha\beta)^2}\right]$$

$$= \frac{y_0(A+\alpha)}{A+\beta A+\alpha\beta}\left[\frac{A(1+\alpha+\alpha\beta)+\alpha^2\beta}{A(1+\beta+\alpha\beta+\alpha\beta^2)+(\alpha\beta)^2}\right] \tag{15}$$

From equations (12)-(15) , the equations (5) and (7) hold at n = 2 .

Now suppose that equations(4-7) hold for n = k . This means that

$$x_{2k-1} = \frac{x_{-1}}{A+\alpha}\prod_{i=1}^{k-1}\left[\frac{A(1+\beta)\left\{\sum_{j=0}^{i-1}(\alpha\beta)^j\right\}+(\alpha\beta)^i}{A(1+\alpha)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(1-A)\left(\alpha^{i+1}\beta^i\right)}\right] \tag{16}$$

$$x_{2k} = \frac{x_0(B+\beta)}{B+\alpha B+\alpha\beta}\prod_{i=1}^{k-1}\left[\frac{B(1+\beta)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(1-B)\left(\alpha^i\beta^{i+1}\right)}{B(1+\alpha)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(\alpha\beta)^{i+1}}\right] \tag{17}$$

$$y_{2k-1} = \frac{y_{-1}}{B+\beta}\prod_{i=1}^{k-1}\left[\frac{B(1+\alpha)\left\{\sum_{j=0}^{i-1}(\alpha\beta)^j\right\}+(\alpha\beta)^i}{B(1+\beta)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(1-B)\left(\beta^{i+1}\alpha^i\right)}\right] \tag{18}$$

$$y_{2k} = \frac{y_0(A+\alpha)}{A+\beta A+\alpha\beta}\prod_{i=1}^{k-1}\left[\frac{A(1+\alpha)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(1-A)\left(\beta^i\alpha^{i+1}\right)}{A(1+\beta)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(\alpha\beta)^{i+1}}\right] \tag{19}$$

Now we will try to prove that equations(4-7) hold at n = k + 1 .

$$x_{2(k+1)-1} = x_{2k+1} = \frac{x_{2k-1}}{x_{2k-1}y_{2k}+\alpha}$$

$$
=\frac{\left[\dfrac{x_{-1}}{A+\alpha}\displaystyle\prod_{i=1}^{k-1}\left[\dfrac{A(1+\beta)\left\{\sum_{j=0}^{i-1}(\alpha\beta)^j\right\}+(\alpha\beta)^i}{A(1+\alpha)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(1-A)\left(\alpha^{i+1}\beta^i\right)}\right]\right]}{\left[\dfrac{x_{-1}}{A+\alpha}\displaystyle\prod_{i=1}^{k-1}\left[\dfrac{A(1+\beta)\left\{\sum_{j=0}^{i-1}(\alpha\beta)^j\right\}+(\alpha\beta)^i}{A(1+\alpha)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(1-A)\left(\alpha^{i+1}\beta^i\right)}\right]\right]\left[\dfrac{y_0(A+\alpha)}{A+\beta A+\alpha\beta}\displaystyle\prod_{i=1}^{k-1}\left[\dfrac{A(1+\alpha)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(1-A)\left(\beta^i\alpha^{i+1}\right)}{A(1+\beta)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(\alpha\beta)^{i+1}}\right]\right]+\alpha}
$$

$$
=\frac{\dfrac{x_{-1}}{A+\alpha}\displaystyle\prod_{i=1}^{k-1}\left[\dfrac{A(1+\beta)\left\{\sum_{j=0}^{i-1}(\alpha\beta)^j\right\}+(\alpha\beta)^i}{A(1+\alpha)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(1-A)\left(\alpha^{i+1}\beta^i\right)}\right]}{\dfrac{A}{A+\beta A+\alpha\beta}\displaystyle\prod_{i=1}^{k-1}\left[\dfrac{A(1+\beta)\left\{\sum_{j=0}^{i-1}(\alpha\beta)^j\right\}+(\alpha\beta)^i}{A(1+\beta)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(\alpha\beta)^{i+1}}\right]+\alpha}
=\frac{\dfrac{x_{-1}\prod_{i=1}^{k-1}\left\{A(1+\beta)\left\{\sum_{j=0}^{i-1}(\alpha\beta)^j\right\}+(\alpha\beta)^i\right\}}{(A+\alpha)\prod_{i=1}^{k-1}\left\{A(1+\alpha)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(1-A)\left(\alpha^{i+1}\beta^i\right)\right\}}}{\left[\dfrac{A\prod_{i=1}^{k-1}\left\{A(1+\beta)\left\{\sum_{j=0}^{i-1}(\alpha\beta)^j\right\}+(\alpha\beta)^i\right\}}{(A+\beta A+\alpha\beta)\prod_{i=1}^{k-1}\left\{A(1+\beta)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(\alpha\beta)^{i+1}\right\}}\right]+\alpha}
$$

$$
=\frac{\left[x_{-1}\prod_{i=1}^{k-1}\left\{A(1+\beta)\left\{\sum_{j=0}^{i-1}(\alpha\beta)^j\right\}+(\alpha\beta)^i\right\}\right]\left[(A+\beta A+\alpha\beta)\prod_{i=1}^{k-1}\left\{A(1+\beta)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(\alpha\beta)^{i+1}\right\}\right]}{\left\{(A+\alpha)\prod_{i=1}^{k-1}\left\{A(1+\alpha)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(1-A)\left(\alpha^{i+1}\beta^i\right)\right\}\right\}\left(\left[A\prod_{i=1}^{k-1}\left\{A(1+\beta)\left\{\sum_{j=0}^{i-1}(\alpha\beta)^j\right\}+(\alpha\beta)^i\right\}\right]+\alpha\left[(A+\beta A+\alpha\beta)\prod_{i=1}^{k-1}\left\{A(1+\beta)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(\alpha\beta)^{i+1}\right\}\right]\right)}
$$

$$
=\frac{\left[\prod_{i=1}^{k-1}\left\{A(1+\beta)\left\{\sum_{j=0}^{i-1}(\alpha\beta)^j\right\}+(\alpha\beta)^i\right\}\right]}{\left\{A\left[A(1+\beta)+\alpha\beta\right]+\alpha(A+\beta A+\alpha\beta)\left(A(1+\beta)\left[\sum_{j=0}^{k-1}(\alpha\beta)^j\right]+(\alpha\beta)^k\right)\right\}}
$$

$$
\times\frac{x_{-1}}{A+\alpha}\frac{\left[(A+\beta A+\alpha\beta)\prod_{i=1}^{k-1}\left\{A(1+\beta)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(\alpha\beta)^{i+1}\right\}\right]}{\left\{\prod_{i=1}^{k-1}\left\{A(1+\alpha)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(1-A)\left(\alpha^{i+1}\beta^i\right)\right\}\right\}\left\{\prod_{i=1}^{k-2}\left\{A(1+\beta)\left[\sum_{j=0}^{i}(\alpha\beta)^j\right]+(\alpha\beta)^{i+1}\right\}\right\}}
$$

$$
=\frac{x_{-1}\prod_{i=1}^{k}\left\{A(1+\beta)\left\{\sum_{j=0}^{i-1}(\alpha\beta)^j\right\}+(\alpha\beta)^i\right\}}{(A+\alpha)\prod_{i=1}^{k}\left\{A(1+\alpha)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(1-A)\left(\alpha^{i+1}\beta^i\right)\right\}}
$$

So

$$x_{2k+1}=\frac{x_{-1}}{(A+\alpha)}\prod_{i=1}^{k}\frac{\left\{A(1+\beta)\left\{\sum_{j=0}^{i-1}(\alpha\beta)^{j}\right\}+(\alpha\beta)^{i}\right\}}{\left\{A(1+\alpha)\left\{\sum_{j=0}^{i}(\alpha\beta)^{j}\right\}+(1-A)\left(\alpha^{i+1}\beta^{i}\right)\right\}} \tag{20}$$

Similarly, by using equations (16-19) we can prove that

$$y_{2(k+1)-1}=y_{2k+1}=\frac{y_{2k-1}}{y_{2k-1}x_{2k}+\beta}$$

$$=\frac{y_{-1}}{(B+\beta)}\prod_{i=1}^{k}\frac{\left\{B(1+\alpha)\left\{\sum_{j=0}^{i-1}(\alpha\beta)^{j}\right\}+(\alpha\beta)^{i}\right\}}{\left\{B(1+\beta)\left\{\sum_{j=0}^{i}(\alpha\beta)^{j}\right\}+(1-B)\left(\beta^{i+1}\alpha^{i}\right)\right\}} \tag{21}$$

By using equations (16-19) and equations (20) , (21) we can prove that

$$x_{2k+2}=\frac{x_{2k}}{x_{2k}y_{2k+1}+\alpha}$$

$$=\frac{x_{0}(B+\beta)}{B+\alpha B+\alpha\beta}\prod_{i=1}^{k}\left[\frac{B(1+\beta)\left\{\sum_{j=0}^{i}(\alpha\beta)^{j}\right\}+(1-B)\left(\alpha^{i}\beta^{i+1}\right)}{B(1+\alpha)\left\{\sum_{j=0}^{i}(\alpha\beta)^{j}\right\}+(\alpha\beta)^{i+1}}\right]$$

$$y_{2k+2}=\frac{y_{2k}}{y_{2k}x_{2k+1}+\beta}$$

$$=\frac{y_{0}(A+\alpha)}{A+\beta A+\alpha\beta}\prod_{i=1}^{k}\left[\frac{A(1+\alpha)\left\{\sum_{j=0}^{i}(\alpha\beta)^{j}\right\}+(1-A)\left(\beta^{i}\alpha^{i+1}\right)}{A(1+\beta)\left\{\sum_{j=0}^{i}(\alpha\beta)^{j}\right\}+(\alpha\beta)^{i+1}}\right]$$

Hence we have finished the proof .

Lemma 2.2: *We have the following relations between the solutions in equations(4-7)*

$$(i)\ x_{2n-1}y_{2n}=\frac{A}{(\alpha\beta)^{n}+A(1+\beta)\sum_{j=0}^{n-1}(\alpha\beta)^{j}}$$

$$(ii)\ y_{2n-1}x_{2n}=\frac{B}{(\alpha\beta)^{n}+B(1+\alpha)\sum_{j=0}^{n-1}(\alpha\beta)^{j}}$$

$$(iii)\ \frac{A}{x_{2n-1}y_{2n}}-\frac{B}{y_{2n-1}x_{2n}}=\left\{A(1+\beta)-B(1+\alpha)\right\}\sum_{j=0}^{n-1}(\alpha\beta)^{j}$$

Proof

(i) From Equation(4) and Equation (7)we have

$$x_{2n-1}y_{2n}=\left\{\frac{x_{-1}}{A+\alpha}\prod_{i=1}^{n-1}\left[\frac{A(1+\beta)\left\{\sum_{j=0}^{i-1}(\alpha\beta)^j\right\}+(\alpha\beta)^i}{A(1+\alpha)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(1-A)\left(\alpha^{i+1}\beta^i\right)}\right]\right\}\left\{\frac{y_0(A+\alpha)}{A+\beta A+\alpha\beta}\prod_{i=1}^{n-1}\left[\frac{A(1+\alpha)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(1-A)\left(\beta^i\alpha^{i+1}\right)}{A(1+\beta)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(\alpha\beta)^{i+1}}\right]\right\}$$

$$=\frac{x_{-1}y_0}{(A+\beta A+\alpha\beta)}\prod_{i=1}^{n-1}\left[\frac{A(1+\beta)\left\{\sum_{j=0}^{i-1}(\alpha\beta)^j\right\}+(\alpha\beta)^i}{A(1+\beta)\left\{\sum_{j=0}^{i}(\alpha\beta)^j\right\}+(\alpha\beta)^{i+1}}\right]=\frac{A(A+\beta A+\alpha\beta)}{(A+\beta A+\alpha\beta)\left\{(\alpha\beta)^n+A(1+\beta)\sum_{j=0}^{n-1}(\alpha\beta)^j\right\}}$$

$$=\frac{A}{(\alpha\beta)^n+A(1+\beta)\sum_{j=0}^{n-1}(\alpha\beta)^j}$$

(ii) As in (i)

(iii) By easy calculations from (i) and (ii) .

Remark 2.3: *We note that*

$$\left|\frac{x_{2n-1}x_{2n}y_{2n-1}y_{2n}}{Ax_{2n}y_{2n-1}-By_{2n}x_{2n-1}}\right|\to 0 \quad as\ \ n\to\infty$$

Lemma 2.4: *If* $A=B=0$ *, We have the following relations*

$(i)\ x_{2n-1}y_{2n}=0 \qquad (ii)\ x_{2n}y_{2n-1}=0$

Theorem 2.5: *We have the following properties for the solution of system (1):*

(i) *If* $A=0$ *and* α *be a positive integer, then* $\lim_{n\to\infty} x_{2n-1}=0$

(ii) *If* $A=0$ *and* β *be a positive integer, then* $\lim_{n\to\infty} y_{2n}=0$

(iii) *If* $B=0$ *and* α *be a positive integer, then* $\lim_{n\to\infty} x_{2n}=0$

(iv) *If* $B=0$ *and* β *be a positive integer, then* $\lim_{n\to\infty} y_{2n-1}=0$

3 Stability of the Solutions of Systems

Stability theory of difference equations and systems of difference equations has attracted many researchers. In recent years there has been much research activity concerning with the global asymptotic stability of system of difference equations. For these stability results, we refer, for example, to [45]. In this section we study the stability of the solutions for systems existed in the previous section and their generalizations. In the beginning, we present the basic notations and definitions concerning with the stability of equilibrium points of systems.

Consider the following two-dimensional system in the form

$$\begin{aligned} x_{n+1}&=f(x_n,y_n)\\ y_{n+1}&=g(x_n,y_n) \end{aligned} \qquad (22)$$

We shall assume that the functions f and g are continuously differentiable.

Definition 3.1: *An equilibrium point of system (22) is a point* $E=(\bar{x},\bar{y})$ *that satisfies*

$$\bar{x}=f(\bar{x},\bar{y})$$
$$\bar{y}=g(\bar{x},\bar{y})$$

Recall the Literalized Stability Theorem for two-dimensional systems in the following proposition(see[13],[28]).

Proposition 3.2: (**Two-Dimensional Version of linearized Stability Theorem**)

Let $F=(f,g)$ *be a continuously differentiable function defined on an open set* W *in* R^2*.Let* $(\bar{x},\bar{x})$ *in* W *be a fixed point of* F *.*

⥣a⇒ If all the eigenvalues of the Jacobian matrix $JF(\bar{x},\bar{y})$ *have modulus less than one, then the equilibrium point* $E=(\bar{x},\bar{y})$ *of system(22) is asymptotically stable.*

⥣b⇒If at least one of the eigenvalues of the Jacobian matrix $JF(\bar{x},\bar{y})$ *has modulus greater than one, then the equilibrium point* $E=(\bar{x},\bar{y})$ *of system(22) is unstable.*

⥣c⇒ The equilibrium point $E=(\bar{x},\bar{y})$ *of system(22) is locally asymptotically stable if every solution of the characteristic equation*

$$\begin{vmatrix} \lambda-\left(\frac{\partial f}{\partial x}(E)\right) & -\frac{\partial f}{\partial y}(E) \\ -\frac{\partial g}{\partial x}(E) & \lambda-\left(\frac{\partial g}{\partial y}(E)\right) \end{vmatrix}=0$$

lies inside the unit circle.

Lemma 3.3: *System(1) has only one equilibrium point which is (0,0) .*

Proof

For the equilibrium points of System(1) , we can write

$$\bar{x}=\frac{\bar{x}}{\bar{x}\,\bar{y}+\alpha} \quad \& \quad \bar{y}=\frac{\bar{y}}{\bar{x}\,\bar{y}+\beta}$$

Then, by solving these equations together, we have the only one equilibrium point which is 0 .

Consider $f(x,y)=\frac{x}{xy+\alpha}$ and $g(x,y)=\frac{y}{xy+\beta}$. The Jacobian of f,g with respect to x,y is given by

$$Jac\left(\frac{f,g}{x,y}\right)=\begin{bmatrix} \frac{\alpha}{(xy+\alpha)^2} & \frac{-x^2}{(xy+\alpha)^2} \\ \frac{-y^2}{(xy+\beta)^2} & \frac{\beta}{(xy+\beta)^2} \end{bmatrix}$$

At the equilibrium point $E=(0,0)$ we have

$$Jac\left(\frac{f,g}{x,y}\right)(E)=\begin{bmatrix} \frac{1}{\alpha} & 0 \\ 0 & \frac{1}{\beta} \end{bmatrix}$$

The corresponding eigenvalues of the equilibrium point E are $\lambda=\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

Thus we have the following theorem

Theorem 3.4: *For system(1) , we have the following cases :*

a) If $|\alpha|<1$ or $|\beta|<1$ *then the equilibrium point E of system(1) is unstable.*

b) *If* $|\alpha|>1$ *or* $|\beta|>1$ *then the equilibrium point E of system(1) is asymptotically stable.*

Now we recall some notations and previous results which will be useful in our study.

Definition 3.5: *The difference equation*

$$x_{n+1}=F\left(x_n,x_{n-1},.....,x_{n-k}\right) \quad , n=0,1,..... \tag{23}$$

is said to be persistence if there exist numbers m *and* M *with* $0<m\le M<\infty$ *such that for any initial conditions* $x_{-k},x_{-k+1},.....,x_{-1},x_0\in(0,\infty)$ *there exists a positive integer* N *which depends on the initial conditions such that* $m\le x_n\le M$ for all $n\ge N$.

Definition 3.6: (i) *The equilibrium point of Equation(23) is locally stable if for every* $\varepsilon>0$ *, there exists* $\delta>0$ *such that for all* $x_{-k},x_{-k+1},.....,x_{-1},x_0\in I$ *with*

$$|x_{-k}-\bar{x}|+|x_{-k+1}-\bar{x}|+....+|x_0-\bar{x}|<\delta$$

we have $|x_n-\bar{x}|<\varepsilon$ *for all* $n\ge -k$.

(ii) *The equilibrium point* $\bar{x}$ *Equation(23) is locally asymptotically stable if* $\bar{x}$ *is locally stable solution of Equation(23) and there exists* $\gamma>0$, *such that for all* $x_{-k},x_{-k+1},.....,x_{-1},x_0\in I$ *with*

$$|x_{-k}-\bar{x}|+|x_{-k+1}-\bar{x}|+....+|x_0-\bar{x}|<\gamma$$

we have $\lim_{n\to\infty} x_n=\bar{x}$.

The literalized equation of Equation(23) about the equilibrium $\bar{x}$ is the linear difference equation

$$y_{n+1}=\sum_{i=0}^{k}\frac{\partial F\left(\bar{x},\bar{x},....,\bar{x}\right)}{\partial x_{n-i}}y_{n-i} \quad .$$

Theorem 3.7: *Assume that* $p,q\in R$ *. Then*

$$|p|+|q|<1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1}-px_n-qx_{n-1}=0 \qquad ,n=0,1,2,...... .$$

---

4 <u>Some Special Cases:</u>

4.1 Case 1 :The Difference Equation (24)

If we consider the one-dimensional case of the system (1) , we have the following generalized difference equation:

$$x_{n+1}=\frac{x_{n-1}}{x_{n-1}x_n+\alpha} \tag{24}$$

Cinar [10] studied the difference equation

$$x_{n+1}=\frac{a\,x_{n-1}}{1+b\,x_{n-1}x_n}$$

If we put $a=b=\frac{1}{\alpha}$, we have the equation (24) . We can obtain the solution for difference equation (24) from theorem(2.1) as special case in the following theorem .

Theorem 4.1.1: *Suppose that $\{x_n\}$ be a solution of equation (24) where α is real number with the initial conditions x_{-1} and x_0 such that $x_{-1}x_0 \neq -\alpha$. Then the solutions of equation (24) have the form*

$$x_1 = \frac{x_{-1}}{A+\alpha} \quad , \; x_2 = \frac{x_0(A+\alpha)}{A(1+\alpha)+\alpha^2}$$

$$x_{2n-1} = \frac{x_{-1}}{A+\alpha}\prod_{i=1}^{n-1}\left[\frac{A(1+\alpha)\left\{\sum_{j=0}^{i-1}(\alpha)^{2j}\right\}+(\alpha)^{2i}}{A(1+\alpha)\left\{\sum_{j=0}^{i}(\alpha)^{2j}\right\}+(1-A)\left(\alpha^{2i+1}\right)}\right]$$

$$x_{2n} = \frac{x_0(A+\alpha)}{A+\alpha A+\alpha^2}\prod_{i=1}^{n-1}\left[\frac{A(1+\alpha)\left\{\sum_{j=0}^{i}(\alpha)^{2j}\right\}+(1-A)\left(\alpha^{2i+1}\right)}{A(1+\alpha)\left\{\sum_{j=0}^{i}(\alpha)^{2j}\right\}+(\alpha)^{2(i+1)}}\right]$$

where $A = x_{-1}x_0$ such that $A \neq \alpha$, $A \neq \dfrac{-\alpha^2}{1+\alpha}$ and $n \geq 2$.

Proof

We can use the mathematical induction to prove the theorem .

Lemma 4.1.2: *We have the following relation between the solutions of equation (24)*

$$x_{2n-1}x_{2n} = \frac{A}{(\alpha)^{2n} + A(1+\alpha)\sum_{j=0}^{n-1}(\alpha)^{2j}}$$

Lemma 4.1.3: *If $A = 0$, then $x_{2n-1}x_{2n} = 0$.*

Now we give the equilibrium points of *equation* (24).

Lemma 4.1.4: *The equilibrium points of the difference equation (24) are 0 and $\pm\sqrt{1-\alpha}$.*

Proof

$$\bar{x} = \frac{\bar{x}}{\bar{x}^2+\alpha}$$

$$\bar{x}\left(\bar{x}^2+\alpha\right) = \bar{x}$$

$$\bar{x}\left(\bar{x}^2+\alpha-1\right) = 0$$

Thus the equilibrium points of the difference equation (24) are 0 and $\pm\sqrt{1-\alpha}$.

Remark 4.1.5: *When $\alpha = 1$, then the only equilibrium point of the difference equation (24) is 0 .*

Theorem 4.1.6: *a) If $|\alpha| < 1$, then the equilibrium point 0 of difference equation (24) is unstable.*

b)If $|\alpha| > 1$, then the equilibrium point 0 of difference equation (24) is asymptotically stable .

c) The equilibrium points $\bar{x} = \pm\sqrt{1-\alpha}$ of difference equation (24) are unstable .

Proof

Let $f : (0,\infty)^2 \to (0,\infty)$ be a continuous function defined by

$$f(u,v) = \frac{u}{uv+\alpha}$$

Therefore it follows that

$$\frac{\partial f(u,v)}{\partial u}=\frac{\alpha}{(uv+\alpha)^2},\ \frac{\partial f(u,v)}{\partial v}=\frac{-u^2}{(uv+\alpha)^2}$$

a) At the equilibrium point $\bar{x}=0$ we have

$$\frac{\partial f(u,v)}{\partial u}=\frac{1}{\alpha}=p_1,\ \frac{\partial f(u,v)}{\partial v}=0=p_2$$

Then the linearized equation of equation (24) about $\bar{x}=0$ is

$$y_{n+1}-p_1y_n-p_2y_{n-1}=0 \qquad ,n=0,1,2,......$$

i. e. $y_{n+1}-\frac{1}{\alpha}y_n=0$ Whose characteristic equation is $\lambda^2-\frac{1}{\alpha}\lambda=0$.

By theorem (3.7) we have $\left|\frac{1}{\alpha}\right|+|0|<1$. Hence $|\alpha|>1$.

b) Obvious .

c) We will prove the theorem at the equilibrium point $\bar{x}=\sqrt{1-\alpha}$ and the proof at the equilibrium point $\bar{x}=-\sqrt{1-\alpha}$ by the same way. At the equilibrium point $\bar{x}=\sqrt{1-\alpha}$ we have

$$\frac{\partial f(u,v)}{\partial u}=\alpha=p_3,\ \frac{\partial f(u,v)}{\partial v}=\alpha-1=p_4$$

Then the linearized equation of equation (24) about $\bar{x}=\sqrt{1-\alpha}$ is

$$y_{n+1}-p_3y_n-p_4y_{n-1}=0 \qquad ,n=0,1,2,......$$

i.e. $y_{n+1}-\alpha y_n-(1-\alpha)y_{n-1}=0 \qquad ,n=0,1,2,......$.

Whose characteristic equation is

$$\lambda^2-\alpha\lambda-(\alpha-1)=0$$

By theorem(3.7) , we have $|\alpha|+|\alpha-1|<1$ which gives the proof .

Here we study the boundedness of equation (24) .

Theorem 4.1.7: *Every solution of* equation*(24) is bounded from above if* $\alpha>0$.

Proof

Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(24). It follows from Eq.(24) that $x_{n+1}=\frac{x_{n-1}}{x_nx_{n-1}+\alpha}\le\frac{1}{x_n}$

Then $x_n\le\frac{1}{x_{n-1}}$ for all $n\ge 0$. This means that every solution of Eq(24) is bouneded from above by $M=\max\left(\frac{1}{x_0},\frac{1}{x_{-1}}\right)$.

4.2 Case 2 :The System (25)

If we consider $\alpha=\beta=1$ in ⇑1⇒, we have the following generalized system of difference equations:

$$x_{n+1}=\frac{x_{n-1}}{y_nx_{n-1}+1}\ \&\ y_{n+1}=\frac{y_{n-1}}{x_ny_{n-1}+1} \tag{25}$$

The system in (25) existed in [30] .We can obtain the solution for this system from theorem(2.1) as special case in the following theorem .

Theorem 4.2.1: *Suppose that* $\{x_n,y_n\}$ *be a solution of system (25) with the initial conditions* x_{-1},x_0,y_{-1} *and* y_0 *such that* $x_0y_{-1}\ne-1$ *and* $x_{-1}y_0\ne-1$. *Then the solutions of system (25) have the form*

$$x_1=\frac{x_{-1}}{A+1},y_1=\frac{y_{-1}}{B+1},x_2=\frac{x_0(B+1)}{2B+1},y_2=\frac{y_0(A+1)}{2A+1}$$

$$x_{2n-1}=x_{-1}\prod_{i=0}^{n-1}\left[\frac{2iA+1}{(2i+1)A+1}\right] \quad \& \quad x_{2n}=x_0\prod_{i=0}^{n-1}\left[\frac{(2i+1)B+1}{2(i+1)B+1}\right]$$

$$y_{2n-1}=y_{-1}\prod_{i=0}^{n-1}\left[\frac{2iB+1}{(2i+1)B+1}\right] \quad \& \quad y_{2n}=y_0\prod_{i=0}^{n-1}\left[\frac{(2i+1)A+1}{2(i+1)A+1}\right]$$

where $A=x_{-1}y_0$, $B=y_{-1}x_0$ *such that* $A\neq-1$, $A\neq\frac{-1}{2}$, $B\neq-1$, $B\neq\frac{-1}{2}$, $B\neq\frac{-1}{2}$ *and* $n\geq2$.

Lemma 4.2.2: *We have the following relations between the solutions in (25)*

(i) $x_{2n-1}y_{2n}=\dfrac{A}{1+2nA}$ (ii) $y_{2n-1}x_{2n}=\dfrac{B}{1+2nB}$

(iii) $\dfrac{A}{x_{2n-1}y_{2n}}-\dfrac{B}{y_{2n-1}x_{2n}}=2n\{A-B\}$

Remarks 4.2.3:

(i) *We note that* $|x_{2n-1}y_{2n}|\to0$ *as* $n\to\infty$.

(ii) *We note that* $|y_{2n-1}x_{2n}|\to0$ *as* $n\to\infty$.

(iii) *We note that* $\left|\dfrac{A}{x_{2n-1}y_{2n}}-\dfrac{B}{y_{2n-1}x_{2n}}\right|\to0$ *as* $n\to0$.

4.3 Case 3 : The Difference Equation (26)

If we consider the one-dimensional case of the system (1) with $\alpha=1$, we have the following difference equation :

$$x_{n+1}=\frac{x_{n-1}}{x_nx_{n-1}+1} \tag{26}$$

This difference equation is eq. (24) with $\alpha=1$. This difference equation was considered in [8] and [42] .We can obtain the solution for difference equation (26) from theorem(2.1) as special case in the following theorem .

Theorem 4.3.1: *Suppose that* $\{x_n\}$ *be a solution of difference equation (26) with the initial conditions* x_{-1} *and* x_0 *such that* $x_0x_{-1}\neq-1$ *. Then the solutions of equation (26) have the form*

$$x_1=\frac{x_{-1}}{A+1} \quad , \quad x_2=\frac{x_0(A+1)}{2A+1}$$

$$x_{2n-1}=x_{-1}\prod_{i=0}^{n-1}\left[\frac{2iA+1}{(2i+1)A+1}\right] \quad \& \quad x_{2n}=x_0\prod_{i=0}^{n-1}\left[\frac{(2i+1)A+1}{2(i+1)A+1}\right]$$

where $A=x_{-1}x_0$ *, such that* $A\neq-1$, $A\neq\frac{-1}{2}$*, and* $n\geq2$.

Proof

We can use the mathematical induction to prove the theorem.

Lemma 4.3.2: *We have the following relation between the solutions of eq. (26)*

$$x_{2n-1}x_{2n}=\frac{A}{1+2nA}$$

Remarks 4.3.3: *We note that* $\left|x_{2n-1}x_{2n}\right|\to 0$ *as* $n\to\infty$.

Theorem 4.3.4: *If* $A=x_{-1}x_0=0$ *, then difference equation(26) is periodic of period two .*

4.4 Case 4 : The Difference Equation (27)

If we consider the one-dimensional case of the system (1) with $\alpha=-1$, we have the following difference equation:

$$x_{n+1}=\frac{x_{n-1}}{x_n x_{n-1}-1} \qquad (27)$$

This difference equation was investigated in [9] .We can obtain the solution for difference equation(27) from theorem(2.1) as special case in the following theorem .

Theorem 4.4.1: *Suppose that* $\{x_n\}$ *be a solution of difference equation (27) with the initial conditions* x_{-1} *and* x_0 *such that* $x_0x_{-1}\neq 1$ *. Then the solutions of equation (27) have the form*

$$x_1=\frac{x_{-1}}{A-1} \quad , \quad x_2=x_0(A-1)$$

$$x_{2n-1}=\frac{x_{-1}}{(A-1)^{n+1}} \quad \& \quad x_{2n}=x_0(A-1)^{n+1}$$

where $A=x_{-1}x_0$ *, such that* $A\neq 1$ *and* $n\geq 2$ *.*

Proof

We can use the mathematical induction to prove the theorem .

Lemma 4.4.2: *We have the following relation between the solutions of eq. (27) :* $x_{2n-1}x_{2n}=A$.

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5. Numerical Examples:

In order to illustrate the results of the previous sections and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to nonlinear difference equations and system of nonlinear difference equations .

Example (5.1)

Consider the difference equation (26) with the initial conditions $x(-1)=1, x(0)=3$. (See Figure 1)

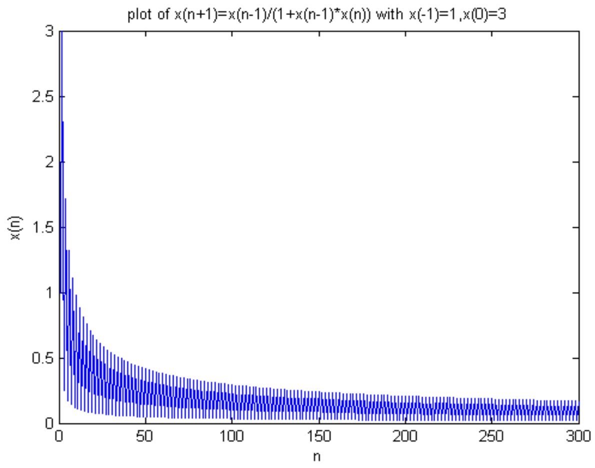

Fig ⥣1⇒ eventually periodic behavior

Example (5.2)

Consider the difference equation (27) with the initial conditions $x(-1)=1, x(0)=3$. (See Figure 2)

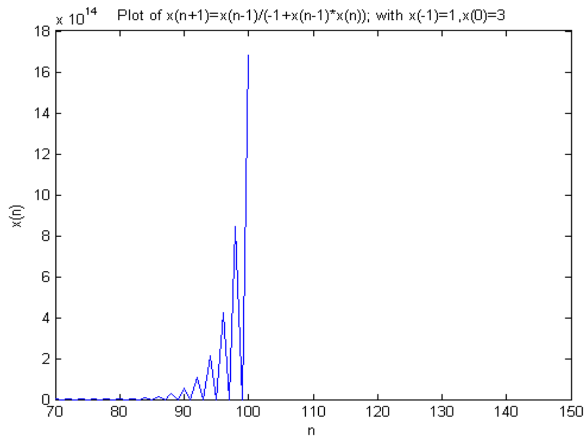

Fig ⥣2⇒ divergence behavior

Example (5.3)

Consider the difference equation system (1) with $\alpha=2, \beta=4$ and the initial conditions $x(-1)=1, x(0)=3$, $y(-1)=2, y(0)=4$. (See Figure 3)

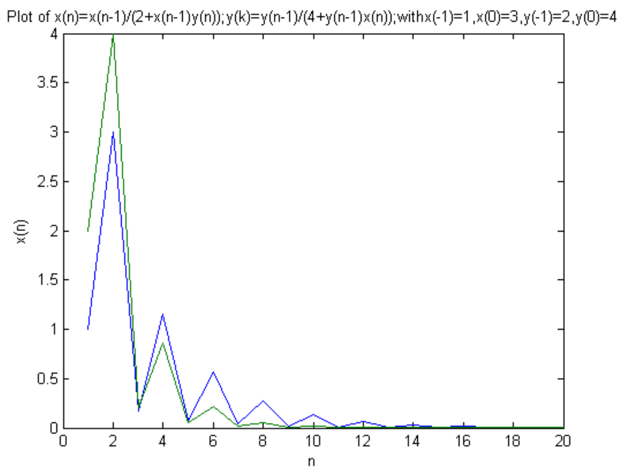

Fig ⥣3⇒ convergence behavior

Conclusion :

We have already investigated the closed form solutions of the two-dimensional fractional system of nonlinear difference equations in the modeling competitive populations in the form (1) . We studied some special cases in one and two dimensional cases whose recently appeared in some recent papers .

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