Large Deformation of Transversely Isotropic Elastic Thin Circular Disk in Rotation

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Abstract-- **This study investigated the large (finite) deformation of transversely isotropic elastic thin circular disk under the influence of rotation. Pertinent energy function is highlighted and the resulting one –dimensional equation of elasticity has been solved. The obtained result showed a negation to the hypothesis of plane section for plane stress problems of elasticity as recorded by linear (small/ infinitesimal deformation) theory of elasticity. The work showed the need to adopt large deformation approach to problems of continuum mechanics.**

Index Term-- **Circular disk, large deformation, rotation, transversely isotropic.**

1. INTRODUCTION

This paper is concerned with large deformation of rotating thin circular disk made of transversely isotropic material. Large deformation consideration of problems of elasticity is often not easy to solve. This essentially is as a result of nonlinearity of the process modeled (large/finite strain modeled). However, large deformation approach to problems of continuum mechanics enables one to obtain vital information which small (infinitesimal) deformation theory might have failed to apprehend. There are many applications of rotating disks in science and engineering. As typical examples, we mention, steam and gas turbines, rotors, and flywheels. In the design of modern structures, increasing use is being made of materials which are transversely isotropic. The analysis of stress distribution in the circular disk rotating is important for a better understanding of the behavior and optimum design of structures. In the context of small deformation theory, the solutions for this problem of rotating disks made of isotropic material can be found in the most standard text books [1-4]. Eraslan and Orcan [5] solved the problem of deformation of a rotating solid disk of exponentially varying thickness, Han and Kai-Yuan [6] considered plastic and elastic stresses for high-speed rotating disks made of isotropic inhomogeneous material with variable thickness, Gupta and Pankaj [7], investigated the thermo elastic-plastic transition in a thin rotating disk with inclusion, Sharma and Sahni [8], solved creep transition problem of transversely isotropic thick-walled rotaing cylinder, Wang and Yin [9] investigated large deformation of elastic half rings, Abd-Alla et al [10], studied the effect of rotation on the radial vibrations in a non-homogeneous orthotropic hollow cylinder while Batra and Iaccarino [11], gave exact solutions for radial deformations of a functionally graded isotropic and incompressible second-order elastic cylinder.

In this work, we considered the radial deformation of a transversely isotropic elastic circular thin disk in the context of large (finite) deformation using semi-linear material. For this we highlight pertinent energy function [12-13] and invoke the concept of hyperelasticity [14]. We take the Frechet derivative [15-16] with respect to the energy conjugate geometry of deformation which is the gradient tensor of position vector in the current configuration. This enables us to obtain constitutive and consequently the equilibrium equation. On this basis, the problem of deformation of thin circular disk in rotation is investigated and the effect of large (finite) deformation would be obtained.

2. PROBLEM SETTING

2.1-Statement of the problem

Let Ω be a transversely isotropic elastic circular thin disk, a subset of three

dimensional Euclidean space E^3 (i.e $\Omega \subset E^3$). We investigate the radial deformation of the thin disk from reference configuration Ω_0 onto current configuration Ω under the influence of rotation:

$$R = R(r),$$

$$\varphi = \theta,$$

where (r,θ) and (R,φ) are the material coordinates in reference Ω_0 and current Ω configurations respectively.

2.2 Geometry of deformation

Let the geometry of deformation of thin disk from reference configuration Ω_0 denoted by the radius vector $\vec{r} = r\vec{e}_r$ onto current configuration Ω denoted by radius vector $\vec{R} = R(r)\vec{e}_r$ be the gradient tensor $\overset{0}{\nabla}\vec{R}$, where $\overset{0}{\nabla}$ is the gradient operator in the reference configuration Ω_0.

Invoking the concept of differentiation in orthogonal curvilinear coordinates and scale factors we obtain the base vectors:

$$\vec{R}_1 = R'(r)\vec{e}_r,$$

$$\vec{R}_2 = R(r)\vec{e}_r,$$

$\vec{r}^1 = \vec{r}_1 = \vec{e}_r$, $\vec{r}^2 = \frac{1}{r}\vec{e}_\theta$, $\vec{r}_2 = r\vec{e}_\theta$,

where $\vec{r}_m, \vec{r}^m, m = 1,2$ are the respective covariant, contravariant basis vectors in the Ω_0 and $\vec{R}_m$ are the covariant basis vectors in Ω.

The gradient tensor $\overset{0}{\nabla}\vec{R}$ is:

$$\overset{0}{\nabla}\vec{R} = \begin{bmatrix} \frac{d}{dr}R & 0 & 0 \\ 0 & \frac{R}{r} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \overset{0}{\nabla}\vec{R}^T. \quad (1)$$

The decomposition of the tensor gradient $\overset{0}{\nabla}\vec{R}$ into stretch symmetric tensor $\tilde{U}$ and orthogonal rotation tensor $\tilde{O}$ is:

$$\overset{0}{\nabla}R = \tilde{U} \cdot \tilde{O}. \quad (2)$$

From equation (2), we obtain the orthogonal rotation tensor $\tilde{O}$:

$$\tilde{O} = \tilde{U}^{-1} \cdot \overset{0}{\nabla}R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (3)$$

where $\tilde{U}^2 = \overset{0}{\nabla}\vec{R}\cdot$

Here and elsewhere, for any vectors $\vec{A}$ and $\vec{B}$, we denote their dot product and cross product respectively by $\vec{A}\cdot\vec{B}$ and $\vec{A}\times\vec{B}$. While for any tensors $\tilde{X}$ and $\tilde{Y}$, we represent their dot product, double dot product and cross product respectively by $\tilde{X}\cdot\tilde{Y}$, $\tilde{X}\cdot\cdot\tilde{Y}$ and $\tilde{X}\times\tilde{Y}$. R' denotes the differentiation of function $R(r)$ with respect to radius r.

3.2-Energy function and equilibrium equation

In the context of small (infinitesimal) deformation, the energy function for isotropic elastic body is [2]:

$$W = \frac{1}{2}\lambda I_1^2(\tilde{\varepsilon}) + \mu I_1(\tilde{\varepsilon})^2, \quad (4)$$

where λ, μ are the lament constants and $I_1(\tilde{\varepsilon}) = \tilde{E}\cdot\cdot\tilde{\varepsilon}$ is the first invariant of the small strain tensor $\tilde{\varepsilon}$.

In large deformation using semi-linear material of John [12] the energy functions for an isotropic and transversely-isotropic bodies are respectively given as[12,13]:

$$W = \frac{1}{2}\lambda S_1^2 + \mu S_2, \quad (5)$$

$$W = \frac{1}{2}\lambda_1 S_1^2 + \lambda_2 S_2 + \lambda_0 S_0, \quad (6)$$

where S_1, S_2 are the invariants of the deformation geometry,

$S_1 = \tilde{E}\cdot\cdot(\tilde{U} - \tilde{E}) \equiv I_1(\tilde{U} - \tilde{E})$,

$S_2 = I_1(\tilde{U} - \tilde{E})^2 = \tilde{E}\cdot\cdot(\tilde{U} - \tilde{E})^2$ and $S_0 = \vec{c}\cdot\tilde{U}^2\cdot\vec{c}$

is an additional invariant of the deformation due to anisotropy, $\vec{c}$ is the unit vector characterizing the direction of anisotropy and $\lambda_0, \lambda_1, \lambda_2$ are material constants. In the case of randomly unidirectional fibre reinforced composite the material constants are the effective moduli [13]:

$$\lambda_2 = \langle \mu \rangle,$$

$$\lambda_1 = \langle \lambda \rangle + \frac{\left\langle \lambda/(\lambda + 2\mu) \right\rangle}{\left\langle 1/(\lambda + 2\mu) \right\rangle} - \left\langle \frac{\lambda^2}{\lambda + 2\mu} \right\rangle,$$

$$\lambda_0 = 2(\lambda_3 - \lambda_2),$$

$$\lambda_3 = \frac{1}{\left\langle 1/\mu \right\rangle},$$

and we note that in the case of degeneracy into isotropic, the energy function (6) reduces to (5) and according for the effective modulli $\lambda_1, \lambda_2, \lambda_3$ while λ_0 vanishes, i.e.

$$\lambda_2 = \lambda_3 = \mu, \quad \lambda_1 = \lambda, \quad \lambda_0 = 0.$$

For any finite function $\chi(\xi, t) \in \Omega \times [0, T)$, $\langle \chi \rangle$ denotes its geometric average over Ω with volume $|\Omega|$:

$$\langle \chi \rangle = \left(\frac{1}{|\Omega|}\right)\int_\Omega \chi d\Omega.$$

Now, invoking the hypothesis of hyperelasticity of Cauchy-Truesdell [14], we take the Frechet derivative [15,16] of the energy function (6) with respect to the geometry of deformation $\overset{0}{\nabla}\vec{R}$ and obtain the Piola's stress tensor $\tilde{P}$, to which it is energy conjugate:

$$\tilde{P} \equiv \frac{\partial W}{\partial \overset{0}{\nabla}\vec{R}},$$

$$\tilde{P} = 2\lambda_2 \overset{0}{\nabla} \vec{R} + (\lambda_1 S_1 - 2\lambda_2)\tilde{O} + \lambda_0 \vec{c}\vec{c} \cdot \overset{0}{\nabla} \vec{R}.$$
(7)

Equation (7) is the constitutive equation for the medium. Under the influence of rotation the equilibrium equation is:

$$\overset{0}{\nabla} \cdot \tilde{P} + \vec{F} = 0,$$
(8)

where the magnitude of the body force $|\vec{F}| = \rho\omega^2 r$, ρ is the mass density of the body, r is the radius of the thin disk and ω is the angular speed of the disk in rotation.

3. RESULTS

The physical components of the stress tensor are obtained through the Cauchy stress tensor $\tilde{T}$ [13]:

$$\tilde{T} = \sqrt{\frac{g}{G}} \overset{0}{\nabla} \vec{R} \cdot \tilde{P},$$
(9)

where in this problem, $\sqrt{\frac{g}{G}} = \frac{r}{RR'}$.

Substituting equations (1) and (3) in equation (7) and setting $\vec{c} = \vec{e}_r$ we have:

$$\tilde{P} = \begin{bmatrix} P_{rr} & 0 & 0 \\ 0 & P_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
(10)

$$P_{rr} = 2(\lambda_2 + \lambda_0)\frac{dR}{dr} + \lambda_1 S_1 - 2\lambda_2,$$
(11)

$$P_{\theta\theta} = 2\lambda_2 \frac{R}{r} + \lambda_1 S_1 - 2\lambda_2,$$
(12)

where $S_1 = \frac{dR}{dr} + \frac{R}{r} - 2$.

Substituting equations (1) and (10) in equation (9) gives:

$$\tilde{T} = \begin{bmatrix} T_{rr} & 0 & 0 \\ 0 & T_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{r}{RR'} \begin{bmatrix} P_{rr}\frac{dR}{dr} & 0 & 0 \\ 0 & \frac{R}{r}P_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$R' = \frac{dR}{dr}.$$
(13)

This implies that:

$$T_{rr} = \frac{r}{R}P_{rr} = \frac{r}{R}\left(2(\lambda_2 + \lambda_0)\frac{dR}{dr} + \lambda_1(\frac{dR}{dr} + \frac{R}{r} - 2) - 2\lambda_2\right),$$
(14)

$$T_{\theta\theta} = \frac{1}{R'}P_{\theta\theta} = \frac{1}{R'}\left(2\lambda_2\frac{R}{r} + \lambda_1(\frac{dR}{dr} + \frac{R}{r} - 2) - 2\lambda_2\right).$$
(15)

The term $2\lambda_0 \frac{r}{R}\frac{dR}{dr}$ in radial component T_{rr} denotes the effect of transversal isotropy.

When the body becomes isotropic $\lambda_0 = 0, \lambda_1 = \lambda, \lambda_2 = \mu$, the components T_{rr} and $T_{\theta\theta}$ are:

$$T_{rr} = \frac{r}{R}P_{rr} = \frac{r}{R}\left(2\mu\frac{dR}{dr} + \lambda(\frac{dR}{dr} + \frac{R}{r} - 2) - 2\mu\right),$$
(16)

$$T_{\theta\theta} = \frac{1}{R'}P_{\theta\theta} = \frac{1}{R'}\left(2\mu\frac{R}{r} + \lambda(\frac{dR}{dr} + \frac{R}{r} - 2) - 2\mu\right).$$
(17)

Using the above expressions of T_{rr} and $T_{\theta\theta}$ we establish the relation:

$$\frac{\frac{R}{r}T_{rr} + T_{\theta\theta}\frac{dR}{dr}}{(\lambda + \mu)} = K,$$
(18)

where K is the volume strain.

Meanwhile in the context of small (infinitesimal) deformation such a relation is [1]:

$$\frac{T_{rr} + T_{\theta\theta}}{(\lambda + \mu)} = K.$$
(19)

Comparing equations (18) and (19) we obtain that equation (18) shows a negation to the hyphothesis of plane section as reported by small deformation theory in equation (19). This indicates the effects of large deformation.

To obtain explicit solution we return to the equilibrium equation (8). The non-vanishing component of equation (8) is:

$$\frac{dP_{rr}}{dr} + \frac{P_{rr} - P_{\theta\theta}}{r} + \rho\omega^2 r = 0.$$
(20)

Substituting equations (11) and (12) in equation (20) we have:

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}-\beta^2\frac{R}{r^2}+\alpha\omega^2 r=0\,,$$
(21)

where $\beta^2=\dfrac{(\lambda_1+\lambda_2)}{(\lambda_1+2\lambda_2+2\lambda_0)}$ and

$$\alpha=\frac{\rho}{(\lambda_1+2\lambda_2+\lambda_0)}.$$

The solution $R=R(r)$ which satisfies equation (21) is:

$$R=R(r)=Ar^{\beta}+\frac{B}{r^{\beta}}-\frac{\alpha\omega^2}{9-\beta^2}r^3,$$
(22)

A and B are the constants of integration.
The displacement $u=u(r)$ of points of the body is:

$$u=u(r)=R-r=Ar^{\beta}+\frac{B}{r^{\beta}}-\frac{\alpha\omega^2}{9-\beta^2}r^3-r\,,$$
(23)

Finally, we make use of boundary conditions for the determination of integration constants A and B:

4. BOUNDARY CONDITIONS

Solid disk with center fixed and outer surface free:
In this case, we obtain the integration constants A and B for the boundary conditions, which specify that the center of the solid thin circular disk is fixed and the outer surface is stress free. The disk is assumed to have radius a.

$$u(r)=0\,,\qquad r=0\,,$$
(24)

$$T_{rr}=0\,,\qquad r=a\,.$$
(25)

Substituting equations (24) and (25) in equations (16) and (23) we obtain:

$$B=0\,,\qquad \text{and}$$

$$A=\left[\frac{2(\lambda_1+\lambda_2)+\dfrac{\alpha\omega^2\lambda_1 a^2}{9-\beta^2}+\dfrac{3(\lambda_1+2\lambda_2+2\lambda_0)\alpha\omega^2 a^2}{9-\beta^2}}{(\lambda_1+2\lambda_2+2\lambda_0)\beta a^{\beta-1}+\lambda_1 a^{\beta-1}}\right]$$
.

Hollow disk with fixed surfaces:
In this case, we obtain the integration constants A and B for the boundary condition, which specify that the inner and outer surfaces of the hollow disk are fixed. The inner and outer radii of the thin disk are a and b respectively.

$$u(r)=0\,,\qquad r=a\,,$$
(26)

$$u(r)=0,\qquad r=b\,.$$
(27)

Substituting equations (26) and (27) in equation (23) we obtain:

$$A=\left[\frac{\dfrac{\alpha\omega^2}{9-\beta^2}(b^{3+\beta}-a^{3+\beta})+(b^{1+\beta}-a^{1+\beta})}{b^{2\beta}-a^{2\beta}}\right],\quad \text{and}$$

$$B=\frac{(ab)^{2\beta}\left[\dfrac{\alpha\omega^2}{9-\beta^2}(a^{3-\beta}-b^{3-\beta})+(a^{1-\beta}-b^{1-\beta})\right]}{b^{2\beta}-a^{2\beta}}.$$

5. CONCLUSIONS

The study investigated large (finite) deformation of transversely isotropic elastic thin circular disk under the influence of rotation. We deduced the relation connecting the physical components of Cauchy's stress tensor. The obtained result showed a negation to the hypothesis of plane section for plane stress problem of elastic thin disk as reported by small (infinitesimal) deformation theory of elasticity. The study showed that large deformation approach to elasticity problems provides ample opportunity to reveal important effects that small (infinitesimal) deformation theory might have failed to apprehend.

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