ON CERTAIN CLASSES OF MEROMORPHICALLY P-VALENT FUNCTIONS WITH POSITIVE COEFFICIENTS DEFINED BY LINEAR OPERATOR

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Abstract. A purpose of this paper is to introduce the class $N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$ of meromorphically p-valent functions by using linear operator. We study various properties such as coefficient inequality, growth and distortion theorems, closure theorems, convolution properties, radii of meromorphically p-valent starlikeness and convexity, weighted mean and arithmetic mean.

Keywords :p- Valent, Hadamard product, Meromorphic, Positive coefficients.

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1. Introduction and definitions

Let $\sum_p$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k} z^{p+k}, (a_{p+k} \geq 0; p \in \mathbb{N} = \{1, 2, ...\}), \tag{1}$$

which are analytic and p-valent in the punctured unit disk $^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = -\{0\}$.

The Hadamard product (or convolution) of two functions, f given by (1) and

$$g(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} b_{p+k} z^{p+k}, (b_{p+k} \geq 0),$$

is defined by

$$(f * g)(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k} b_{p+k} z^{p+k}.$$

For real or complex numbers $\alpha_1, \alpha_2, ..., \alpha_q$ and $\beta_1, \beta_2, ..., \beta_s (\beta_j \neq 0, -1, -2, ...; j = 1, 2, ..., s)$, we define the generalized hypergeometric function ${}_qF_s(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z)$ by (see for example, [13, p. 19]),

$${}_qF_s(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k, (\alpha_2)_k, ..., (\alpha_q)_k}{(\beta_1)_k, (\beta_2)_k, ..., (\beta_s)_k (1)_k} z^k$$

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$$(q \leq s+1; s, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in),$$

where $(x)_n$ is the Pochhammer symbol, defined in terms of the gamma function Γ by,

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & (n = 0; x \in \mathbb{C} - \{0\}, \\ x(x+1)(x+2)...(x+n-1) & (n \in \mathbb{N}; x \in \mathbb{C}, \end{cases}$$

corresponding to the function
$h_p(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z) = z^{-p} \, {}_qF_s(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z)$,
Liu and Srivastava [8] (see for details [5] and [6]) introduced a linear operator:

$$H_{p,q,s}(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s) : \sum_p \to \sum_p,$$

which is defined by the Hadamard product:

$$H_{p,q,s}(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s) f(z) = h_p(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z) * f(z),$$

$$(\beta_j \neq 0, -1, -2, ...; j = 1, 2, ..., s, q \leq s+1; s, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in),$$

For notational simplicity, we use

$$H_{p,q,s}(\alpha_1) = H_{p,q,s}(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s).$$

For a function of the form (1), we have

$$H_{p,q,s}(\alpha_1) f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \Gamma_k a_{p+k} z^{p+k},$$

where

$$\Gamma_k = \frac{(\alpha_1)_{p+k}, (\alpha_2)_{p+k}, ..., (\alpha_q)_{p+k}}{(\beta_1)_{p+k}, (\beta_2)_{p+k}, ..., (\beta_s)_{p+k} (1)_{p+k}}$$

Then we can easily verify that:

$$z(H_{p,q,s}(\alpha_1) f(z))' = \alpha_1 H_{p,q,s}(\alpha_1) f(z) - (\alpha_1 + p) H_{p,q,s}(\alpha_1) f(z).$$

The linear operator $H_{p,q,s}(\alpha_1)$ was investigated recently by Liu and Srivastava [9], Aouf [2], and Aouf and Yassen [4]. Now we define the class $N_{p,q,s}(\alpha_1; \alpha, \beta, \gamma)$ as follows:

Definition 1.1 : A function $f(z)$ of the form (1) is said to be in the class $N_{p,q,s}(\alpha_1; \alpha, \beta, \gamma)$ if it satisfies the following inequality:

$$\left| \frac{z^{p+1}(H_{p,q,s}(\alpha_1) f(z))' + p}{(2\gamma - 1) z^{p+1}(H_{p,q,s}(\alpha_1) f(z))' + (2\gamma\alpha - p)} \right| < \beta, \quad (2)$$

where $0 \leq \alpha < p, 0 < \beta \leq 1, \frac{1}{2} \leq \gamma \leq 1, \alpha_1 > 0, p \in \mathbb{N}, q \leq s+1; s, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in^*$.

The following are special classes of the class $N_{p,q,s}(\alpha_1; \alpha, \beta, \gamma)$:

(1) $N_{p,q,s}(\alpha_1; \alpha, 1, 1) = \{f \in \sum_p : \Re\{-z^{p+1}(H_{p,q,s}(\alpha_1) f(z))'\} > \alpha, 0 \leq \alpha < p, \alpha_1 > 0, p \in \mathbb{N}, q \leq s+1; s, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in^*\}$.

(2) $N_{p,2,1}(a,1;c;\alpha,1,1) = \{f \in \sum_p : \Re\{-z^{p+1}(\ell_p(a,c)f(z))' > 0, 0 \le \alpha < p, a, c > 0, p \in \mathbb{N}, z \in^*\}$, see [8].
(3) $N_{p,2,1}(x+p,p;p;\alpha,1,1) = \{f \in \sum_p : \Re\{-z^{p+1}(D^{x+p-1}f(z))' > 0, 0 \le \alpha < p, x > -p, p \in \mathbb{N}, x \in^*\}$, see [1], [3].
(4) $N_{p,2,1}(x,1;x+1;\alpha,1,1) = \{f \in \sum_p : \Re\{-z^{p+1}(F_{x,p}f(z))' > 0, 0 \le \alpha < p, x > 0, p \in \mathbb{N}, x \in^*\}$, see [1], [14], [15].

Meromorphic multivalent functions have been studied (for example) by many outhers such as Rain and Srivastava [11], Yang [15], EL- Ashwah [7], Saif and Kilicman [12], Mostafa [10] and others.

In this paper, we derive several interesting properties for the class $N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$ such as coefficient inequality, growth and distortion theorems, closure theorems, Hadamard properties, radii of meromorphically p-valent starlikeness, convexity and weighted mean and arithmetic mean for these functions.

2. Coefficient Estimates

Theorem 2.1 : A function $f(z)$ defined by (1) is said to be in the class $N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$ if and only if

$$\sum_{k=0}^{\infty}(p+k)(1+2\beta\gamma-\beta)\Gamma_k a_{p+k} \le 2\beta\gamma(p-\alpha), \tag{3}$$

where

$$\Gamma_k = \frac{(\alpha_1)_{p+k},(\alpha_2)_{p+k},...,(\alpha_q)_{p+k}}{(\beta_1)_{p+k},(\beta_2)_{p+k},...,(\beta_s)_{p+k}(1)_{p+k}},$$

and

$0 \le \alpha < p, 0 < \beta \le 1, \frac{1}{2} \le \gamma \le 1, \alpha_1 > 0, p \in \mathbb{N}, q \le s+1; s,q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in^*$.

proof : Assume that (3) holds. Then

$$|z^{p+1}(H_{p,q,s}(\alpha_1)f(z))' + p| - \beta|(2\gamma-1)z^{p+1}(H_{p,q,s}(\alpha_1)f(z))' + (2\gamma\alpha - p)| < 0$$

$$|z^{p+1}(-pz^{-p-1} + \sum_{k=0}^{\infty}(p+k)\Gamma_k a_{p+k}z^{p+k-1}) + p|$$

$$-\beta|(2\gamma-1)z^{p+1}(-pz^{-p-1} + \sum_{k=0}^{\infty}(p+k)\Gamma_k a_{p+k}z^{p+k-1}) + (2\gamma\alpha+p)| < 0$$

$$|\sum_{k=0}^{\infty}(p+k)\Gamma_k a_{p+k}z^{2p+k}| - \beta|2\gamma(p-\alpha) + \sum_{k=0}^{\infty}(p+k)(2\gamma-1)\Gamma_k a_{p+k}z^{2p+k}| < 0.$$

For $|z| = r < 1$

$$\sum_{k=0}^{\infty}(p+k)\Gamma_k a_{p+k}r^{2p+k} - 2\beta\gamma(p-\alpha) + \sum_{k=0}^{\infty}\beta(p+k)(2\gamma-1)\Gamma_k a_{p+k}r^{2p+k}$$

$$< \sum_{k=0}^{\infty}(p+k)(1+2\beta\gamma-\beta)\Gamma_k a_{p+k} - 2\beta\gamma(p-\alpha) \leq 0.$$

Thus $f \in N_{p,q,s}(\alpha_1; \alpha, \beta, \gamma)$.

Conversely, assume that $f \in N_{p,q,s}(\alpha_1; \alpha, \beta, \gamma)$.

Then by (2), we have

$$|\frac{z^{p+1}(H_{p,q,s}(\alpha_1)f(z))' + p}{(2\gamma-1)z^{p+1}(H_{p,q,s}(\alpha_1)f(z))' + (2\gamma\alpha - p)}|, z \in^*$$

$$|\frac{\sum_{k=0}^{\infty}(p+k)\Gamma_k a_{p+k} z^{2p+k}}{2\gamma(p-\gamma) + \sum_{k=0}^{\infty}(p+k)(2\gamma-1)\Gamma_k a_{p+k} z^{2p+k}}| < \beta, z \in^* .$$

Since $|\Re(z)| \leq |z|$ for all z, then

$$\Re\{\frac{\sum_{k=0}^{\infty}(p+k)\Gamma_k a_{p+k} z^{2p+k}}{2\gamma(p-\gamma) + \sum_{k=0}^{\infty}(p+k)(2\gamma-1)\Gamma_k a_{p+k} z^{2p+k}}\} < \beta, z \in^* . \quad (4)$$

Now choosing the values of z on the real axis so that the function $z^{p+1}(H_{p,q,s}(\alpha_1)f(z))'$ is real. By clearing the denominator in (4) and letting $z \to 1^-$ through positive values, we get:

$$\sum_{k=0}^{\infty}(p+k)(1+2\beta\gamma-\beta)\Gamma_k a_{p+k} \leq 2\beta\gamma(p-\alpha)$$

Hence the proof is complete.

Corollary 2.1 Let the function $f(z)$ defined by (1) be in the class $N_{p,q,s}(\alpha_1; \alpha, \beta, \gamma)$. Then

$$a_{p+k} \leq \frac{2\beta\gamma(p-\alpha)}{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}, (k \geq 0, p \in \mathbb{N})$$

The result is sharp for the function:

$$f(z) = \frac{1}{z^p} + \frac{2\beta\gamma(p-\alpha)}{(p+k)(1+2\beta\gamma-\beta)\Gamma_k} z^{p+k}, (k \geq 0, p \in \mathbb{N}) \quad (5)$$

3. Growth and Distortion Theorem

A growth and distortion property for the function f to be in the class $N_{p,q,s}(\alpha_1; \alpha, \beta, \gamma)$ is given as follows:

Theorem 3.1 : Let the function $f(z)$ defined by (1) be in the class $N_{p,q,s}(\alpha_1; \alpha, \beta, \gamma)$. Then for $0 < |z| = r < 1$, we have

$$\frac{1}{r^p} - \frac{2\beta\gamma(p-\alpha)}{p(1+2\beta\gamma-\beta)\Gamma_0} r^p \leq |f(z)| \leq \frac{1}{r^p} + \frac{2\beta\gamma(p-\alpha)}{p(1+2\beta\gamma-\beta)\Gamma_0} r^p, \quad (6)$$

and

$$\frac{p}{r^{p+1}} - \frac{2\beta\gamma(p-\alpha)}{(1+2\beta\gamma-\beta)\Gamma_0} r^{p-1} \leq |f'(z)| \leq \frac{p}{r^{p+1}} + \frac{2\beta\gamma(p-\alpha)}{(1+2\beta\gamma-\beta)\Gamma_0} r^{p-1}, \quad (7)$$

with equality for

$$f(z) = \frac{1}{z^p} + \frac{2\beta\gamma(p-\alpha)}{p(1+2\beta\gamma-\beta)\Gamma_0} z^p (p \in \mathbb{N}) \quad (8)$$

proof : By Theorem 2.1, we have

$$p(1+2\beta\gamma-\beta)\Gamma_0\sum_{k=0}^{\infty}a_{p+k}\leq\sum_{k=0}^{\infty}(p+k)(1+2\beta\gamma-\beta)\Gamma_k a_{p+k}\leq 2\beta\gamma(p-\alpha).$$

Then

$$\sum_{k=0}^{\infty}a_{p+k}\leq\frac{2\beta\gamma(p-\alpha)}{p(1+2\beta\gamma-\beta)\Gamma_0}$$

for $0<|z|=r<1$,

$$|f(z)|\leq\frac{1}{r^p}+\sum_{k=0}^{\infty}a_{p+k}r^{p+k},$$

$$\leq\frac{1}{r^p}+r^p\sum_{k=0}^{\infty}a_{p+k}$$

$$\leq\frac{1}{r^p}+\frac{2\beta\gamma(p-\alpha)}{p(1+2\beta\gamma-\beta)\Gamma_0}r^p$$

and

$$|f(z)|\geq\frac{1}{r^p}-\sum_{k=0}^{\infty}a_{p+k}r^{p+k},$$

$$\geq\frac{1}{r^p}-r^p\sum_{k=0}^{\infty}a_{p+k},$$

$$\geq\frac{1}{r^p}-\frac{2\beta\gamma(p-\alpha)}{p(1+2\beta\gamma-\beta)\Gamma_0}r^p$$

which, together, yield (6). Also from Theorem 2. 1, it follows that

$$\sum_{k=0}^{\infty}(p+k)a_{p+k}\leq\frac{2\beta\gamma(p-\alpha)}{(1+2\beta\gamma-\beta)\Gamma_0}.$$

Thus

$$|f'(z)|\leq|\frac{-p}{z^{p+1}}|+\sum_{k=0}^{\infty}(p+k)a_{p+k}z^{p+k-1},$$

$$|f'(z)|\leq\frac{p}{r^{p+1}}+r^{p-1}\sum_{k=0}^{\infty}(p+k)a_{p+k}$$

$$\leq\frac{p}{r^{p+1}}+\frac{2\beta\gamma(p-\alpha)}{(1+2\beta\gamma-\beta)\Gamma_0}r^{p-1},$$

and

$$|f'(z)|\geq|\frac{-p}{z^{p+1}}|-\sum_{k=0}^{\infty}(p+k)a_{p+k}z^{p+k-1},$$

$$|f'(z)|\geq\frac{p}{r^{p+1}}-r^{p-1}\sum_{k=0}^{\infty}(p+k)a_{p+k}$$

$$\geq\frac{p}{r^{p+1}}-\frac{2\beta\gamma(p-\alpha)}{(1+2\beta\gamma-\beta)\Gamma_0}r^{p-1},$$

which, together, yield (7).
It is clear that the function given by (8) is extremal function.
Hence the proof is complete.

4. closure Theorems

Theorem 4.1 : Let

$$f_{p-1}(z)=\frac{1}{z^p},$$

and

$$f_{p+k}(z)=\frac{1}{z^p}+\frac{2\beta\gamma(p-\alpha)}{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}z^{p+k},(k\geq 0,p\in\mathbb{N}).$$

Then $f(z)$ is in the class $N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$ if and only if it can be expressed in the form

$$f(z)=\sum_{k=-1}^{\infty}\lambda_{p+k}f_{p+k}(z),$$

where $\lambda_{p+k}\geq 0$ and $\sum_{k=-1}^{\infty}\lambda_{p+k}=1$.

proof : First suppose that $f(z)$ can be expressed of the form

$$f(z)=\sum_{k=-1}^{\infty}\lambda_{p+k}f_{p+k}(z)$$

$$=\frac{1}{z^p}+\sum_{k=0}^{\infty}\frac{2\beta\gamma(p-\alpha)\lambda_{p+k}}{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}z^{p+k}$$

Then

$$\sum_{k=0}^{\infty}\frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)}\frac{2\beta\gamma(p-\alpha)\lambda_{p+k}}{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}$$

$$=\sum_{k=0}^{\infty}\lambda_{p+k}=1-\lambda_{p-1}\leq 1.$$

Hence $f\in N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$.
Conversely, suppose that $f\in N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$, then

$$a_{p+k}\leq\frac{2\beta\gamma(p-\alpha)}{(p+k)(1+2\beta\gamma-\beta)\Gamma_k},(k\geq 0,p\in\mathbb{N})$$

Setting

$$\lambda_{p+k}=\frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)}a_{p+k}$$

and

$$\lambda_{p-1}=1-\sum_{k=0}^{\infty}\lambda_{p+k},$$

we get

$$f(z)=\sum_{k=-1}^{\infty}\lambda_{p+k}f_{p+k}(z)$$

Hence the proof is complete.

Theorem 4.2 : Let the functions $f_i(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,i} z^{p+k}, i = 1, 2, ..., n$ be in the class $N_{p,q,s}(\alpha_1; \alpha, \beta, \gamma)$. Then the function
$F(z) = \sum_{i=1}^{n} \lambda_i f_i(z)$, where $\sum_{i=1}^{n} \lambda_i = 1$,
is also in the class $N_{p,q,s}(\alpha_1; \alpha, \beta, \gamma)$.

proof : From Theorem 2.1, we have

$$\sum_{k=0}^{\infty}(p+k)(1+2\beta\gamma-\beta)\Gamma_k a_{p+k} \leq 2\beta\gamma(p-\alpha).$$

Since

$$F(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty}(\sum_{i=1}^{n} \lambda_i a_{p+k,i}) z^{p+k}$$

Then

$$\sum_{k=0}^{\infty}(p+k)(1+2\beta\gamma-\beta)\Gamma_k(\sum_{i=1}^{n} \lambda_i a_{p+k,i})$$

$$= \sum_{i=1}^{n} \lambda_i \sum_{k=0}^{\infty}(p+k)(1+2\beta\gamma-\beta)\Gamma_k a_{p+k,i}$$

$$\leq 2\beta\gamma(p-\alpha)\sum_{i=1}^{n} \lambda_i = 2\beta\gamma(p-\alpha).$$

This completes the proof of the theorem.

Theorem 4.3 : The class $N_{p,q,s}(\alpha_1; \alpha, \beta, \gamma)$ is convex.

proof : In order to proof the theorem it is enough to show that the function $h(z)$ defined by

$$h(z) = \lambda f(z) + (1-\lambda)g(z), (0 \leq \lambda \leq 1)$$

is in the class $N_{p,q,s}(\alpha_1; \alpha, \beta, \gamma)$,
where

$$f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k} z^{p+k}, a_{p+k} \geq 0$$

$$g(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} b_{p+k} z^{p+k}, b_{p+k} \geq 0,$$

are in the class $N_{p,q,s}(\alpha_1; \alpha, \beta, \gamma)$.
Then

$$h(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty}(\lambda a_{p+k} + (1-\lambda)b_{p+k}) z^{p+k}.$$

By using Theorem 2. 1, we get

$$\sum_{k=0}^{\infty}(p+k)(1+2\beta\gamma-\beta)\Gamma_k(\lambda a_{p+k} + (1-\lambda)b_{p+k})$$

$$\leq \lambda 2\beta\gamma(p-\alpha) + (1-\lambda)2\beta\gamma(p-\alpha)$$

$$= 2\beta\gamma(p-\alpha).$$

Thus $h(z) \in N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$.
This completes the proof of the theorem.

5. Convolution Properties

Theorem 5.1 : Let the functions

$$f_i(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,i} z^{p+k}, (a_{p+k,i} \geq 0; i = 1, 2), \tag{9}$$

be in the class $N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$, then $(f_1 * f_2) \in N_{p,q,s}(\alpha_1;\eta,\beta,\gamma)$, where

$$\eta = p - \frac{2\beta\gamma(p-\alpha)^2}{p(1+2\beta\gamma-\beta)\Gamma_0}.$$

The result is sharp for the functions $f_i(z)(i = 1, 2)$ given by

$$f_i(z) = \frac{1}{z^p} + \frac{2\beta\gamma(p-\alpha)}{p(1+2\beta\gamma-\beta)\Gamma_0} z^p, (i = 1, 2, p \in \mathbb{N})$$

proof : Since $f_i(z) \in N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)(i = 1, 2)$.
Then from Theorem 2.1, we have:

$$\sum_{k=0}^{\infty} \frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)} a_{p+k,i} \leq 1 (i = 1, 2).$$

Thus by the Cauchy- Schwarz inequality, we obtain

$$\sum_{k=0}^{\infty} \frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)} \sqrt{a_{p+k,1}a_{p+k,2}} \leq 1 \tag{10}$$

To prove the theorem we need to find the largest η such that

$$\sum_{k=0}^{\infty} \frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\eta)} a_{p+k,1}a_{p+k,2} \leq 1,$$

or we must get:

$$\frac{a_{p+k,1}a_{p+k,2}}{p-\eta} \leq \frac{\sqrt{a_{p+k,1}a_{p+k,2}}}{p-\alpha} (k \geq 0; p \in \mathbb{N}),$$

which is equivalent to

$$\sqrt{a_{p+k,1}a_{p+k,2}} \leq \frac{p-\eta}{p-\alpha} (k \geq 0; p \in \mathbb{N}).$$

From (10), we have

$$\frac{2\beta\gamma(p-\alpha)}{(p+k)(1+2\beta\gamma-\beta)\Gamma_k} \leq \frac{p-\eta}{p-\alpha}.$$

By simplifying it, we get:

$$\eta \leq p - \frac{2\beta\gamma(p-\alpha)^2}{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}.$$

Now, defining the function $\phi(k)$ by

$$\phi(k) = p - \frac{2\beta\gamma(p-\alpha)^2}{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}.$$

This function is an increasing function of k. Thus, we have

$$\eta \leq \phi(0) = p - \frac{2\beta\gamma(p-\alpha)^2}{p(1+2\beta\gamma-\beta)\Gamma_0}.$$

Hence the proof is complete.

Theorem 5.2 : Let the function $f_1(z)$ defined by (9) be in the class $N_{p,q,s}(\alpha_1;\alpha_2,\beta,\gamma)$ and the function $f_2(z)$ defined by (9) be in the class $N_{p,q,s}(\alpha_1;\alpha_3,\beta,\gamma)$. Then $(f_1 * f_2)(z) \in N_{p,q,s}(\alpha_1;\zeta,\beta,\gamma)$, where

$$\zeta = p - \frac{2\beta\gamma(p-\alpha_2)(p-\alpha_3)}{p(1+2\beta\gamma-\beta)\Gamma_0}$$

The result is sharp for the functions $f_i(z)(i=1,2)$ given by

$$f_1(z) = \frac{1}{z^p} + \frac{2\beta\gamma(p-\alpha_2)}{p(1+2\beta\gamma-\beta)\Gamma_0} z^p (p \in \mathbb{N}),$$

and

$$f_2(z) = \frac{1}{z^p} + \frac{2\beta\gamma(p-\alpha_3)}{p(1+2\beta\gamma-\beta)\Gamma_0} z^p (p \in \mathbb{N}).$$

proof : By using the same technique of Theorem 5.1 we prove the theorem, hence it is omitted.

Theorem 5.3 : Let $f_1(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,1} z^{p+k} \in N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$ and $f_2(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,2} z^{p+k} \in N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$ with $|a_{p+k,2}| \leq 1, k \geq 0, p \in \mathbb{N}$. Then $(f_1 * f_2)(z) \in N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$.

proof : By using Theorem 2.1 it is enough to show that:

$$\sum_{k=0}^{\infty} \frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)} a_{p+k,1} a_{p+k,2} \leq 1,$$

Since

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)} |a_{p+k,1} a_{p+k,2}|, \\
&= \sum_{k=0}^{\infty} \frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)} a_{p+k,1} |a_{p+k,2}|, \\
&\leq \sum_{k=0}^{\infty} \frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)} a_{p+k,1}, \\
&\leq 1.
\end{aligned}$$

Thus $(f_1 * f_2)(z) \in N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$.

This completes the proof of the theorem.

Corollary 5.1 : Let $f_1(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,1} z^{p+k} \in N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$ and $f_2(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,2} z^{p+k} \in N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$ with $0 \le |a_{p+k,2}| \le 1, k \ge 0, p \in \mathbb{N}$. Then $(f_1 * f_2)(z) \in N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$.

Theorem 5.4 : If the functions $f_i(z)(i = 1,2)$ defined by (9) are in the class $N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$ and

$$p(1+2\beta\gamma-\beta)\Gamma_0 - 4\beta\gamma(p-\alpha) \ge 0,$$

then the function $h(z)$ defined by

$$h(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty}(a_{p+k,1}^2 + a_{p+k,2}^2) z^{p+k},$$

is also in the class $N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$.

proof : From Theorem 2.1, we have

$$\sum_{k=0}^{\infty} \frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)} a_{p+k,1} \le 1,$$

and

$$\sum_{k=0}^{\infty} \frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)} a_{p+k,2} \le 1.$$

Then

$$\sum_{k=0}^{\infty} [\frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)}]^2 a_{p+k,1}^2 \le 1,$$

and

$$\sum_{k=0}^{\infty} [\frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)}]^2 a_{p+k,2}^2 \le 1.$$

Hence

$$\sum_{k=0}^{\infty} \frac{1}{2} [\frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)}]^2 (a_{p+k,1}^2 + a_{p+k,2}^2) \le 1.$$

To proof the theorem it is sufficient to show that

$$\sum_{k=0}^{\infty} [\frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)}] (a_{p+k,1}^2 + a_{p+k,2}^2) \le 1. \tag{11}$$

Thus the inequality (11) will be satisfied if

$$\frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)} \le \frac{1}{2}[\frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)}]^2, k = 0,1,2,...$$

or

$$(p+k)(1+2\beta-\beta)\Gamma_k - 4\beta\gamma(p-\alpha) \ge 0, k = 0,1,2,....$$

The left hand side of the above inequality is an increasing function of k, so it satisfied for all k if

$$p(1+2\beta-\beta)\Gamma_0 - 4\beta\gamma(p-\alpha) \ge 0,$$

which is given by our hypothesis.

This completes the proof of the theorem.

Theorem 5.5 : If the functions $f_i(z)(i=1,2)$ defined by

$$f_i(z)=\frac{1}{z^p}+\sum_{k=0}^{\infty}a_{p+k,i}z^{p+k},\,(a_{p+k,i}\geq 0;i=1,2), \tag{8}$$

are in the class $N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$, then the function $h(z)$ defined by

$$h(z)=\frac{1}{z^p}+\sum_{k=0}^{\infty}(a_{p+k,1}^2+a_{p+k,2}^2)z^{p+k},$$

is in the class $N_{p,q,s}(\alpha_1;\zeta,\beta,\gamma)$, where

$$\zeta=p-\frac{4\beta\gamma(p-\alpha)^2}{p(1+2\beta\gamma-\beta)\Gamma_0}$$

The result is sharp for the functions $f_i(z)(i=1,2)$ given by

$$f_i(z)=\frac{1}{z^p}+\frac{2\beta\gamma(p-\alpha)}{p(1+2\beta\gamma-\beta)\Gamma_0}z^p,\,(i=1,2,p\in\mathbb{N})$$

proof : From Theorem 2.1, we have

$$\sum_{k=0}^{\infty}\frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)}a_{p+k,i}\leq 1(i=1,2).$$

Now

$$\sum_{k=0}^{\infty}[\frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)}]^2a_{p+k,i}^2$$

$$\leq\sum_{k=0}^{\infty}[\frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)}a_{p+k,i}]^2\leq 1(i=1,2),$$

hence

$$\sum_{k=0}^{\infty}\frac{1}{2}[\frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)}]^2(a_{p+k,1}^2+a_{p+k,2}^2)\leq 1. \tag{12}$$

To prove the theorem, it is sufficient to show that:

$$\sum_{k=0}^{\infty}\frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\zeta)}(a_{p+k,1}^2+a_{p+k,2}^2)\leq 1. \tag{13}$$

From (12)(13), we must find the largest value of ζ such that

$$\frac{1}{p-\zeta}\leq\frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k}{4\beta\gamma(p-\alpha)^2},$$

or

$$\zeta\leq p-\frac{4\beta\gamma(p-\alpha)^2}{(p+k)(1+2\beta\gamma-\beta)\Gamma_k},$$

since the right hand side of the above inequality is an increasing function of k, we get

$$\zeta=p-\frac{4\beta\gamma(p-\alpha)^2}{p(1+2\beta\gamma-\beta)\Gamma_0}.$$

Hence the proof is complete.

6. Radii of Meromorphically p-Valent Starlikness and Convexity

Theorem 6.1 : Let the function $f(z)$ defined by (1) be in the class $N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$. Then $f(z)$ is meromorphically p-valent starlike of order $\delta(0\leq\delta<p)$ in the disk $|z|<r_1$, where

$$r_1=\inf_{k\geq 0}\{\frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k(p-\delta)}{2\beta\gamma(p-\alpha)(p+k-\delta)}\}^{\frac{1}{2p+k}}$$

The result is sharp .

proof : From Theorem 2. 1, we have:

$$\sum_{k=0}^{\infty}(p+k)(1+2\beta\gamma-\beta)\Gamma_k a_{p+k}\leq 2\beta\gamma(p-\alpha),$$

and $f(z)$ is said to be meromorphically p-valent starlike of order $\delta(0\leq\delta<p)$, if

$$\Re\{-\frac{zf'(z)}{f(z)}\}>\delta,$$

or

$$|\frac{zf'(z)+pf(z)}{f(z)}|\leq p-\delta(0\leq\delta<p).$$

Now

$$|\frac{zf'(z)+pf(z)}{f(z)}|=|\frac{\sum_{k=0}^{\infty}(2p+k)a_{p+k}z^{p+k}}{z^{-p}+\sum_{k=0}^{\infty}a_{p+k}z^{p+k}}|,$$
$$\leq\frac{\sum_{k=0}^{\infty}(2p+k)a_{p+k}|z|^{2p+k}}{1-\sum_{k=0}^{\infty}a_{p+k}|z|^{2p+k}}.$$

To prove the theorem the above inequality must be less than or equal to $p-\delta$, so

$$\sum_{k=0}^{\infty}\frac{(p+k-\delta)}{(p-\delta)}a_{p+k}|z|^{2p+k}\leq p-\delta. \quad (14)$$

Then by Corollary 2. 1 the inequality (14) will be true if

$$|z|^{2p+k}\leq\frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k(p-\delta)}{2\beta\gamma(p-\alpha)(p+k-\delta)},$$

that is,

$$|z|\leq\{\frac{(p+k)(1+2\beta\gamma-\beta)\Gamma_k(p-\delta)}{2\beta\gamma(p-\alpha)(p+k-\delta)}\}^{\frac{1}{2p+k}},$$

The infimum of the above quantity is the radii of starlikeness of the function $f(z)$ in the class $N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$.

The sharpness follows by choosing the same extremal function (5).

This completes the proof of the theorem.

Theorem 6.2 : Let the function $f(z)$ defined by (1) be in the class $N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$. Then $f(z)$ is meromorphically p-valent convex of order $\mu(0\leq\mu<p)$ in the disk $|z|<r_2$, where

$$r_2=\inf_{k\geq 0}\{\frac{p(p-\mu)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)(3p+k-\mu)}\}^{\frac{1}{2p+k}}.$$

The result is sharp .

proof : It is enough to show that

$$\Re\{-1-\frac{zf''(z)}{f'(z)}\} > \mu(0\leq\mu<p); |z|<r_2; p\in\mathbb{N}),$$

or

$$|\frac{(zf'(z))'+pf'(z)}{f'(z)}| = |\frac{\sum_{k=0}^{\infty}(p+k)(2p+k)a_{p+k}z^{p+k}}{-pz^{-(p+1)}+\sum_{k=0}^{\infty}(p+k)a_{p+k}z^{p+k-1}}|,$$

$$\leq\frac{\sum_{k=0}^{\infty}(p+k)(2p+k)a_{p+k}|z|^{2p+k}}{p-\sum_{k=0}^{\infty}(p+k)a_{p+k}|z|^{2p+k}}.$$

To prove the theorem the above inequality must be less than or equal to $p-\mu$, or

$$\sum_{k=0}^{\infty}\frac{(p+k)(3p+k-\mu)}{p(p-\mu)}a_{p+k}|z|^{2p+k}\leq 1.$$

By using Theorem 2.1,we obtain

$$|z|^{2p+k}\leq\frac{p(p-\mu)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)(3p+k-\mu)}.$$

Thus

$$|z|\leq\{\frac{p(p-\mu)(1+2\beta\gamma-\beta)\Gamma_k}{2\beta\gamma(p-\alpha)(3p+k-\mu)}\}^{\frac{1}{2p+k}}(k\geq 0, p\in\mathbb{N}.$$

By choosing r_2 to be the infimum of the above quantity we get the result.
The sharpness follows by choosing the same extremal function (5).
This completes the proof of the theorem.

7. Weighted Mean and Arithmetic Mean

Definition 1.1 : If the functions $f(z)$ and $g(z)$ defined by (1) are in the class $N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$, then the weighted mean $h_i(z)$ of the two functions is defined as follows

$$h_i(z)=\frac{1}{2}[(1-i)f(z)+(1+i)g(z)].$$

Theorem 7.1 : If the functions $f(z)$ and $g(z)$ defined by (1) are in the class $N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$. Then their weighted mean is also in the class $N_{p,q,s}(\alpha_1;\alpha,\beta,\gamma)$.

proof : The weighted mean of $f(z)$ and $g(z)$ is:

$$h_i(z)=\frac{1}{2}[(1-i)f(z)+(1+i)g(z)],$$

$$=\frac{1}{2}[(1-i)(\frac{1}{z^p}+\sum_{k=0}^{\infty}a_{p+k}z^{p+k})+(1+i)(\frac{1}{z^p}+\sum_{k=0}^{\infty}b_{p+k}z^{p+k})],$$

$$=\frac{1}{z^p}+\sum_{k=0}^{\infty}\frac{1}{2}((1-i)a_{p+k}+(1+i)b_{p+k})z^{p+k}.$$

By using Theorem 2.1, it is sufficient to show that

$$\sum_{k=0}^{\infty}(p+k)(1+2\beta\gamma-\beta)\Gamma_k[\frac{1}{2}((1-i)a_{p+k}+(1+i)b_{p+k})],$$

$$= \frac{1}{2}(1-i)\sum_{k=0}^{\infty}(p+k)(1+2\beta\gamma-\beta)\Gamma_k a_{p+k} + \frac{1}{2}(1+i)\sum_{k=0}^{\infty}(p+k)(1+2\beta\gamma-\beta)\Gamma_k b_{p+k},$$

$$\leq \frac{1}{2}(1-i)(2\beta\gamma(p-\alpha)) + \frac{1}{2}(1+i)(2\beta\gamma(p-\alpha)),$$

$$= 2\beta\gamma(p-\alpha).$$

Hence $h_i(z) \in N_{p,q,s}(\alpha_1; \alpha, \beta, \gamma)$.
This completes the proof of the theorem.

Theorem 7.2 : If the functions $f_i(z)(i = 1, ..., d)$ defined by

$$f_i(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,i} z^{p+k}, (a_{p+k,i} \geq 0, k \geq 0, i = 1, ..., d),$$

belongs to the class $N_{p,q,s}(\alpha_1; \alpha, \beta, \gamma)$, then their arithmetic mean defined by

$$h(z) = \frac{1}{d}\sum_{i=1}^{d} f_i(z),$$

is also in the class $N_{p,q,s}(\alpha_1; \alpha, \beta, \gamma)$.

proof : Since

$$h(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty}(\frac{1}{d}\sum_{i=1}^{d} a_{p+k,i})z^{p+k}.$$

Then by using Theorem 2. 1, we must show that

$$\sum_{k=0}^{\infty}(p+k)(1+2\beta\gamma-\beta)\Gamma_k(\frac{1}{d}\sum_{i=1}^{d} a_{p+k,i}),$$

$$= \frac{1}{d}\sum_{i=1}^{d}\sum_{k=0}^{\infty}(p+k)(1+2\beta\gamma-\beta)\Gamma_k a_{p+k,i},$$

$$\leq \frac{1}{d}\sum_{i=1}^{d} 2\beta\gamma(p-\alpha) = 2\beta\gamma(p-\alpha).$$

Therefore $h(z) \in N_{p,q,s}(\alpha_1; \alpha, \beta, \gamma)$.
Hence the proof is complete.

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