Von Neumann Stability Analysis of Reduced Navier-Stokes Equations in Vorticity-Stream Function Formulation

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Abstract

In [1, 2] Fourier Stability Analysis of Navier-Stokes Equations has been carried out,where non-linearity were incorporated by freezing non-linear terms and obtained optimum smoothing factors which were in a close agreement with practical smoothing factors. In this paper an effort was made to obtain Von-Neumann stability of elliptic type reduced Navier-Stokes equations in vorticity-stream function formulation. Using stream function in cartesian coordinate system we derived the Vorticity transport equation by eliminating pressure term from momentum equations in cross differentiation. The derived equation is classified as a parabolic equation with the unknown vorticity ,ω. By considering the definition of the vorticity, we derived the equation known as the stream function equation and is classified as elliptic PDE with the unknown stream function ,ψ. The stability analysis is carried out by converting stream function equation expressed in non-dimensional form so that the non-linearity is vanished in the equation. With the induction of vorticity and stream function,the incompressible Navier-Stokes equations are decoupled into one elliptic and one parabolic equation. The Von Neumann stability analysis is carried out for the elliptic equation only. The various numerical experiments using iterative schemes have been carried out to obtain theoretically optimal smoothing factors. Results are plotted for amplification factors for a number of unknown parameters.

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1 Navier-Stokes Equation

Incompressible and unsteady Navier-Stokes Equations in cartesian forms are as follows:- Continuity Equation:-

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

Momentum Equations without external force term:-

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \quad (2)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) \quad (3)$$

2 Vorticity-Stream Function Formulation

The incompressible Navier-stokes Equations are decoupled into one elliptic equation and and one parabolic equation. For a two-dimensional, incompressible flow,a function may be defined which satisfies the continuity equation. Such a function is known as the stream function and, in cartesian coordinate system,is given by

$$v = -a\frac{\partial \psi}{\partial x}$$

$$u = b\frac{\partial \psi}{\partial y}$$

The Vorticity Equation is

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{4}$$

In order to derive the vorticity transport equation, pressure is eliminated from momentum equations by cross differentiation. Differentiation with respect to y of equation(2) yields

$$\frac{\partial^2 u}{\partial y \partial t} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x \partial y} + \frac{\mu}{\rho}(\frac{\partial^3 u}{\partial y \partial x^2} + \frac{\partial^3 u}{\partial y^3}) \tag{5}$$

whereas the differentiation with respect to x of equation(3) yields

$$\frac{\partial^2 v}{\partial x \partial t} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial x \partial y} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x \partial y} + \frac{\mu}{\rho}(\frac{\partial^3 v}{\partial x^3} + \frac{\partial^3 v}{\partial x \partial y^2}) \tag{6}$$

subtracting equation(6) from equation(5) to obtain [3]

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \frac{\mu}{\rho}(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}) \tag{7}$$

The vorticity equation can be reduced into

$$a\frac{\partial^2 \psi}{\partial x^2} + b\frac{\partial^2 \psi}{\partial y^2} = -\omega \tag{8}$$

3 Non-dimensionalization

Equation(7) and equation(8) can be non-dimensionalized by using the following substitutions
$\Omega = \frac{\omega L}{u_\infty}$, $X = \frac{x}{L}$, $Y = \frac{y}{L}$, $T = \frac{u_\infty t}{L}$
$u = \frac{u}{u_\infty}$, $v = \frac{v}{v_\infty}$,$\frac{1}{Re} = \frac{\mu}{\rho u_\infty L}$, $\psi = \frac{\psi}{u_\infty L}$

Equation(7) implies that

$$\frac{\partial \Omega}{\partial T} + u\frac{\partial \Omega}{\partial X} + v\frac{\partial \Omega}{\partial Y} = \frac{1}{Re}(\frac{\partial^2 \Omega}{\partial X^2} + \frac{\partial^2 \Omega}{\partial Y^2}) \tag{9}$$

Equation(8) implies that

$$a\frac{\partial^2 \psi}{\partial X^2} + b\frac{\partial^2 \psi}{\partial Y^2} = -\Omega \tag{10}$$

Equations (9) and (10) are reduced parabolic and elliptic respectively. For the seek of stability analysis, we consider equation(10) only.

4 Stability Analysis for Stream Function Equation

The Finite Difference Formulation for equation (10) is as follows:-

$$a\psi_{i+1,j} - 2(a+b)\psi_{i,j} + a\psi_{i-1,j} + b\psi_{i,j+1} + b\psi_{i,j-1} = -h^2\Omega_{i,j} \tag{11}$$

By applying Jacobi iterative method, we get

$$a\psi^k_{i+1,j} - 2(a+b)\psi^{k+1}_{i,j} + a\psi^k_{i-1,j} + b\psi^k_{i,j+1} + b\psi^k_{i,j-1} = -h^2\Omega_{i,j} \tag{12}$$

Subtracting equation(12) from equation (11) we get

$$ae^k_{i+1,j} - 2(a+b)e^{k+1}_{i,j} + ae^k_{i-1,j} + be^k_{i,j+1} + be^k_{i,j-1} = 0 \tag{13}$$

Let $e^k_{i,j} = A(\theta, \phi)e^{\iota(i\theta + j\phi)}$
$e^{k+1}_{i,j} = \bar{A}(\theta, \phi)e^{\iota(i\theta + j\phi)}$

After substitution in equation(13) and simplification , we get amplification factor as follows:-
$G(\theta,\phi) = \frac{\bar{A}(\theta,\phi)}{A(\theta,\phi} = \frac{a\cos\theta + b\cos\phi}{a+b}$
The Finite Difference Formulation for equation (10) is as follows:-

$$a\psi_{i+1,j} - 2(a+b)\psi_{i,j} + a\psi_{i-1,j} + b\psi_{i,j+1} + b\psi_{i,j-1} = -h^2\Omega_{i,j} \quad (14)$$

By applying Gauss-Seidel iterative method, we get

$$a\psi^k_{i+1,j} - 2(a+b)\psi^{k+1}_{i,j} + a\psi^{k+1}_{i-1,j} + b\psi^k_{i,j+1} + b\psi^{k+1}_{i,j-1} = -h^2\Omega_{i,j} \quad (15)$$

Subtracting equation(15) from equation (14) we get

$$ae^k_{i+1,j} - 2(a+b)e^{k+1}_{i,j} + ae^{k+1}_{i-1,j} + be^k_{i,j+1} + be^{k+1}_{i,j-1} = 0 \quad (16)$$

Let $e^k_{i,j} = A(\theta,\phi)e^{\iota(i\theta+j\phi)}$ $e^{k+1}_{i,j} = \bar{A}(\theta,\phi)e^{\iota(i\theta+j\phi)}$ After substitution in equation(16) and simplification , we get amplification factor as follows:-

$$G(\theta,\phi) = \frac{\bar{A}(\theta,\phi)}{A(\theta,\phi)} = \frac{ae^{\iota\theta} + be^{\iota\phi}}{2a + 2b - ae^{-\iota\theta} - be^{-\iota\phi}} \quad (17)$$

5 Results and discussions

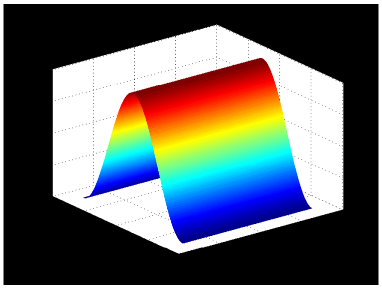
The main purpose of Von Neumann stability analysis [3] has been to assess the smoothing factor of iterative schemes,Jacobi and Guass-Seidel methods for reduced Navier-Stokes equations.The Von Neumann stability analysis was carried out for Stream Function only.In order to analyze non-linear formulation of stream function equation we defined u and v velocities with the multiple of constant coefficient a and b multiplied by differentials of stream function in both x and y coordinates. This made the stream function formulation non-linear in non-dimensional form. Both constants a and b varies between (0,1). Therefore we carried out Von Nuemann stability analysis of non-linear stream function equation and effectively the Navier-Stokes equations. Numerical experiments were carried out with a number of combination of the values of a and b. Figure 1 shows the smoothing factor for various values of a and b and it was found that the maximum smoothing factor for Jacobi Iterative scheme are for a=0 and for all values of b except zero and for all values of a except zero with b=0, and for all other combination of values of a and b lies below the maximum. Figure 2 shows the smoothing factor for various values of a and b and it was also found that the maximum smoothing factor for Gauss-Seidel, although lesser than Jacobi methods behaves in similar way that is for a=0 and for all values of b except zero and for all values of a except zero with b=0. The similar behavior is found for other values of a and b but lies below the maximum. This analysis concluded that if the reduced Navier-Stokes are solved by the described iterative scheme solution is guaranteed. The Von Neumann stability analysis may also be carried out for other formed of reduced Navier-Stokes equations, like vorticity transport formulation similar to this analysis. Further this analysis can also be done for other iterative scheme. A thorough comparative analysis of all iterative scheme will be done in future work.

References

[1] T.M.Shah, N.Talat, Three Dimensional Analysis and Application of A Line Solver for the Recirculating Flows Using Multi-grid Methods. to be published in IJEIT.

[2] T.M.Shah,D.F.Mayers,and J.S.Rollett. Analysis and Application of A Line Solver for the Recirculating Flows Using Multigrid methods. In R.Rannacher W.Hackbusch,editor,Numerical Treatment of the Navier-Stokes Equations,pages 134-144,Braunschweig,January 1990.Vieweg.

[3] Klaus A. Hoffmann and Steve T.Chiang,"Computational Fluid Dynamics".



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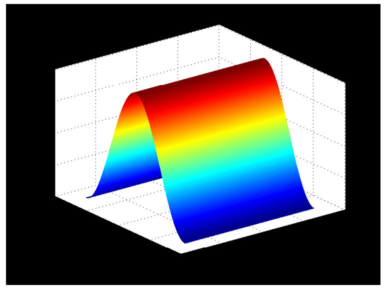

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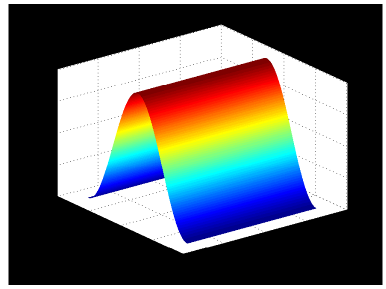

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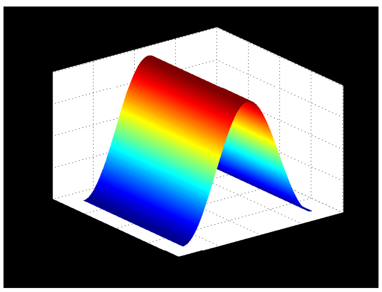

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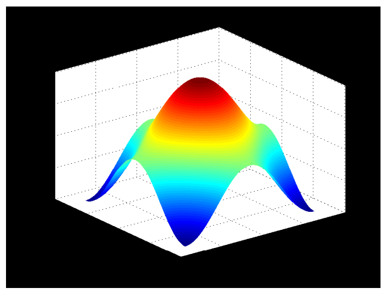

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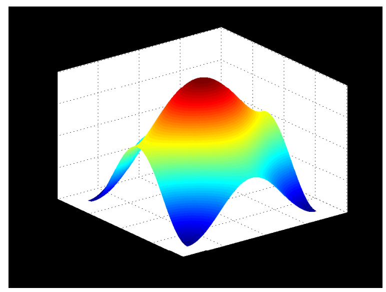

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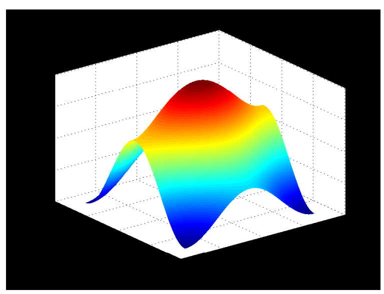
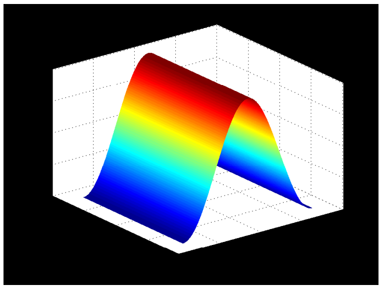


(g) Amplification Factor at a=0.5,b=1.0

Figure 1: Amplification Factor at various (a,b) values for Jacobi method



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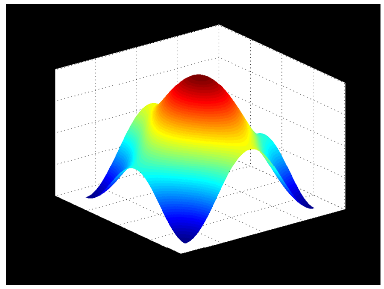

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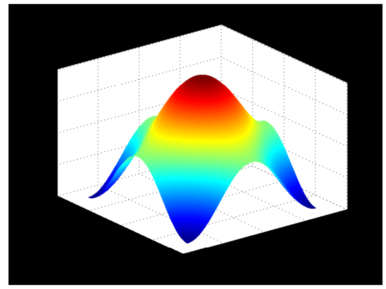

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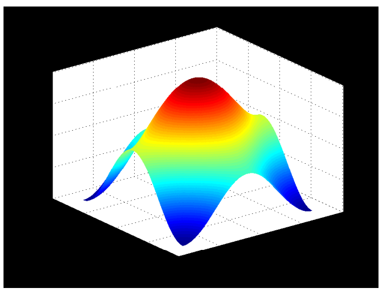

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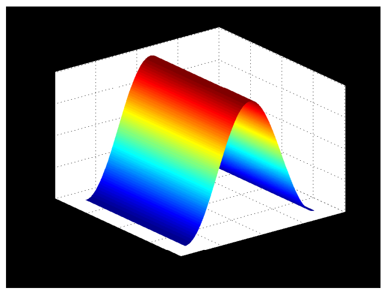

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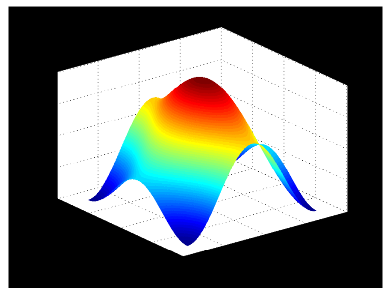

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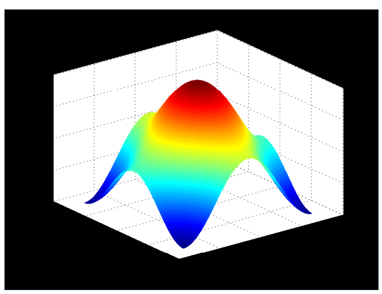

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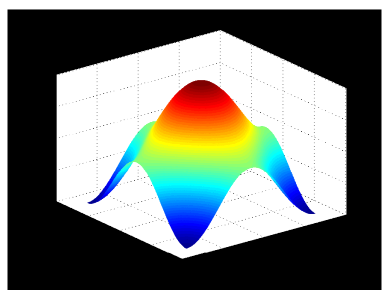
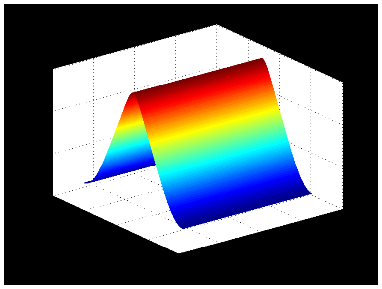


(h) Amplification Factor at a=1,b=1

Figure 2: Amplification Factor at various (a,b) values for Jacobi method



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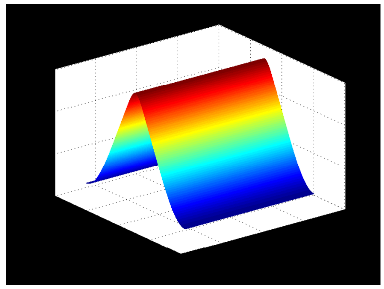

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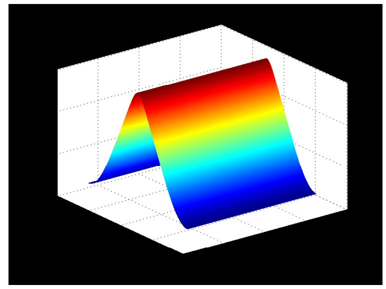

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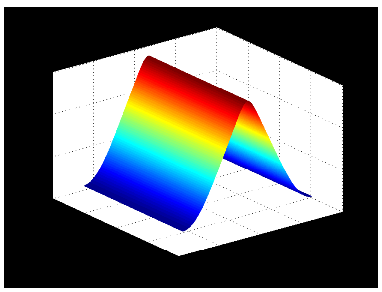

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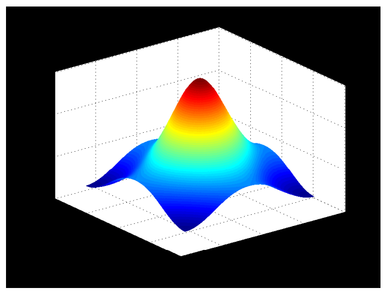

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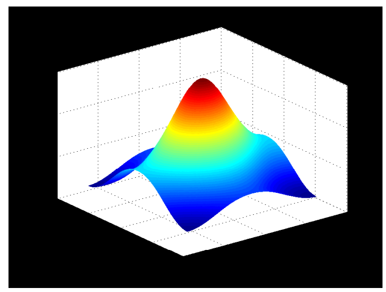

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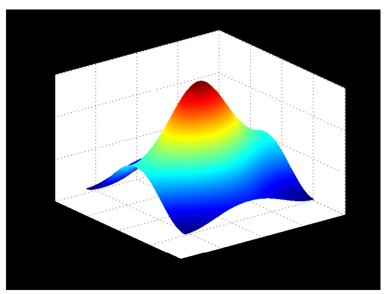
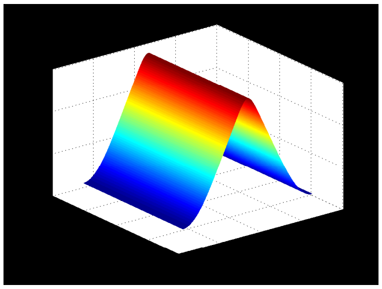


(g) Amplification Factor at a=0.5,b=1

Figure 3: Amplification Factors at various (a,b) values for Gauss-Seidel method



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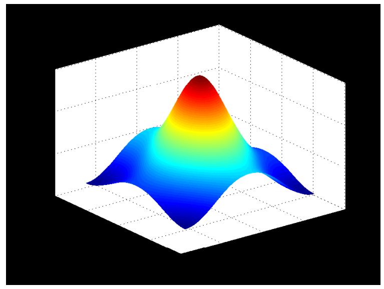

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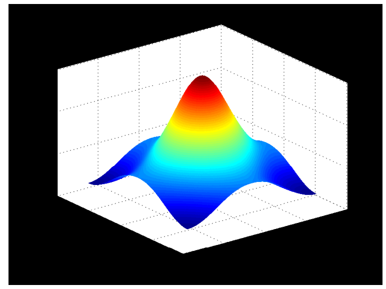

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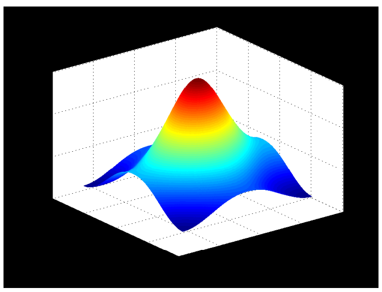

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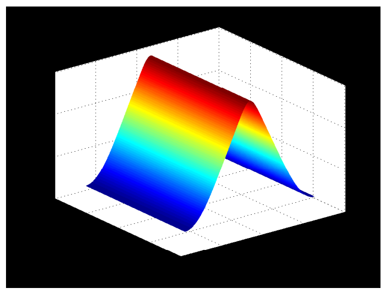

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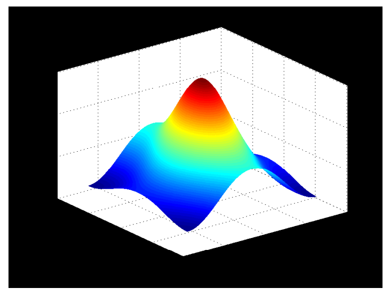

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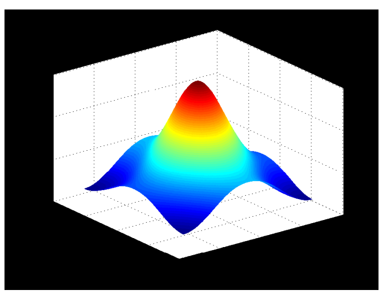

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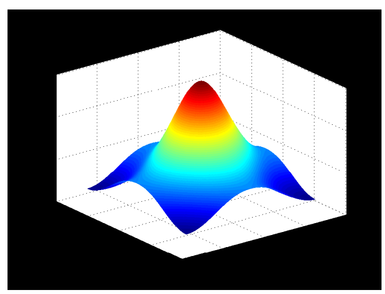

(h) Amplification Factor at a=1,b=1

Figure 4: Amplification Factors at various (a,b) values for Gauss-Seidel method