On group rings with involution

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Abstract. In this note, we consider the involution group ring AG of an involution group G over an involution ring A with identity and prove the involutive version of a Theorem due to Connell, which characterizes *-artinian group rings with identity. Furthermore, *-simple involution group rings are also investigated.

Keywords. Group rings, Involution, biideal, *-artinian, *-noetherian and *-simple.

2000 Mathematics Subject Classification. 16W10, 16S34.

1 Introduction

We consider only associative rings. An *involution ring* A is a ring with *involution* * subject to the identities

$$a^{**} = a, (ab)^* = b^*a^*, (a+b)^* = a^* + b^*,$$

for all $a, b \in A$. Thus, the involution is an anti-isomorphism of order 2 on A.

By an *involution group* we mean a multiplicative group with involution satisfying the first two identities. Every group G has at least one involution, namely the unary operation of taking inverse; that is $g^* = g^{-1}$, for every $g \in G$.

Recall that a *biideal* B of a ring A is a subring of A satisfying $BAB \subseteq B$. An ideal (biideal) I of an involution ring A is called **-ideal* (**-biideal*), if it is closed under involution; that is $I^* = I$.

By the way, in the theory of involution rings, *-biideals have been used successfully (instead of one-sided ideals) in describing their structure (see for instance [1] and [3]).

Recall that a ring A is said to be *simple* if $A^2 \neq 0$ and A has no proper nonzero ideals. Analogously, an involution ring A is called **-simple* if $A^2 \neq 0$ and A has no proper nonzero *-ideals. Obviously, a simple involution ring is *-simple while the converse is not true. For example, if A is a simple ring and A^{op} is its opposite ring then the ring $R = A \oplus A^{op}$, under the exchange involution * defined by $(a, b)^* = (b, a)$ for every $(a, b) \in R$, is *-simple but not simple (see [2]).

Finally, an involution ring A is said to be *-artinian (*-noetherian) if it satisfies dcc (acc) on *-biideals (see [1] and [3]).

The following results due to Beidar and Wiegandt [3] and Aburawash [1] are useful in proving our main result.

Proposition 1 *([3], Theorem 3). If a ring A with involution satisfies dcc on *-biideals, then its Jacobson radical $\Im(A)$ satisfies dcc on subgroups and hence $\Im(A)$ is nilpotent*

Proposition 2 *([3], Corollary 4). A ring A with involution has dcc on *-biideals if and only if A is an artinian ring with artinian Jacobson radical.*

Proposition 3 *([1], Corollary 1). If $A_1, A_2, \ldots.., A_n$ are *-arinian (*-noetherian) involution rings, then so is their direct sum $A_1 \oplus A_2 \oplus \ldots.. \oplus A_n$.*

2 Group rings with involution

Let A be a ring with identity and let G be a group. Following [5], the group ring AG consists of all finite linear combinations $\sum_{g\in G} a_g g$, $a_g \in A$, $g \in G$, where only finitely many coefficients a_g are nonzero. In AG, addition is defined componentwise while multiplication is an extension of that in G. Furthermore, AG has identity and A is a subring of AG.

For Baer radical, Lemma 73.4 in [5] shows that the prime radical $\beta(A)$ of the ring A is a subring of the prime radical $\beta(AG)$ of the group ring AG.

Proposition 4 *([5], Lemma 73.4). If AG is the group ring of a group G over a ring A, then $\beta(A) = A \cap \beta(AG)$.*

Moreover, the following result due to T. Connell gives a necessary and sufficient condition for a group ring with identity to be artinian.

Proposition 5 *([5], Lemma 73.11). Let AG be the group ring of a group G over a ring A with identity. Then AG is artinian if and only if A is artinian and G is finite.*

Now, let A be an involution ring with involution $\triangledown$ and G be a group with involution $\lozenge$. One can make AG an involution ring by defining an involution * on the group ring AG in a natural way by $(\sum_{g\in G} a_g g)^* = \sum_{g\in G} a_g^{\triangledown} g^{\lozenge}$, $a_g \in A$, $g \in G$. However, we can simply denote all the given involutions by *, since it will not lead to ambiguity.

Our main goal is to check whether, for involution rings with identity, descending chain condition on *-biideals is transferred between the group ring and its underlying ring. This is, however, the involutive version of T. Connell's Theorem given in Proposition 4 (see also [4]).

Theorem 6 *Let AG be an involution group ring of a group G over a ring A with identity. Then AG is *-artinian if and only if the ring A is *-artinian and the group G is finite.*

Proof. If AG is *-artinian then it is artinian and $\Im(A)$ satisfies dcc on subgroups, by Proposition 1. Hence, from Proposition 5, it follows that A is artinian and G is finite. A and AG are both artinian, so $\Im(A) = \beta(A)$ and $\Im(AG) = \beta(AG)$, whence by Proposition 4, $\Im(A) \subseteq \Im(AG)$. Thus $\Im(A)$ satisfies dcc on subgroups since $\Im(AG)$ does, and consequently $\Im(A)$ is artinian. Applying Proposition 2, A would be *-artinian. For sufficiency, let A be *-artinian and G be finite. Since AG , as a left A-module, is a direct sum of $|G|$ copies of A, then AG is *-artinian, according to Proposition 3. ■

For acc on *-biideals, we have

Proposition 7 *If the involution ring A is *-noetherian and the group G is finite, then the involution group ring AG is *-noetherian.*

Proof. Let A be *-noetherian and G be finite. Since AG , as a left A-module, is a direct sum of $|G|$ copies of A, then AG is *-noetherian, by Proposition 3. ■

3 *-Simple Group rings with involution

The characterization of *-simple involution rings was given in [2] as follows.

Proposition 8 *([2], Lemma 1).An involution ring A is *-simple if and only if either A is simple or $A = I \oplus I^*$, with $I \lhd A$ and I is a simple ring, and the involution is the exchange involution $\triangledown$ defined by $(a, b^*)^\triangledown = (b, a^*)$, for all $a, b \in I$.*

The following result gives a sufficient and necessary condition for a group ring to be simple.

Proposition 9 *A group ring AG is simple if and only if the ring A is simple and the group $G = \{1\}$.*

Proof. For necessity, let AG be simple. From [5], page 313, we have

$$\omega G = \{x \in AG \mid x = \sum_g a_g g \text{ with} \sum_g a_g = 0\} \lhd AG \text{ and } A \cong AG/\omega G.$$

If $G \neq \{1\}$, then $0 \neq y = a1 - ag \in \omega G$ implies $\omega G = AG$, whence $A = 0$, contradicts $AG \neq 0$. Thus $G = \{1\}$ and consequently $\omega G = 0$. Hence $A \cong AG$ is simple. The sufficiency is obvious. ■

Finally, using Propositions 8 and 9, we get the following characterization of *-simple involution group rings.

Theorem 10 *An involution group ring AG is *-simple if and only if the ring A is *-simple and the group $G = \{1\}$.*

Proof. Let AG be *-simple. Since ωG is a *-ideal of AG, then either $\omega G = AG$ or $\omega G = 0$. The first case is impossible because $A \cong AG/\omega G$ implies $A = 0$, a contradiction. The second case implies, from $A \cong AG/\omega G$, that $A \cong AG$ is

*-simple and consequently $G = \{1\}$. For the converse, Let A be *-simple and $G = \{1\}$, then according to Proposition 8, either A is simple or $A = I \oplus I^*$, with $I \lhd A$ and I is a simple ring. If A is simple then, by Proposition 9, AG is simple and so *-simple. If A is not simple, then the group ring $AG \cong A = I \oplus I^*$ is *-simple. ■

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