Some Computational Aspects of Martingale Processes in ruling the Arbitrage from Binomial asset Pricing Model

Sheik Ahmed Ullah

***Abstract*— This paper concerns about some computational aspect of martingale processes in binomial asset pricing models. *Mathematica* programs are incorporated to get the martingale values, which lead to no-arbitrage option values through first fundamental theorem of asset pricing, for a set of risk neutral probabilities. Numerical example, through *Mathematica* program, ensures the theoretical fact that if not discounted properly the underlying stock price dynamics doesn't follow martingale process.**

***Index Terms*— risk neutral probability, martingale, arbitrage, option price, discounting.**

I. Introduction

Arbitrage plays an important role in the dynamics of a stock market. Before investing money in any particular stock market it is very important to check about its existence. For a proper economy we should always try to rule out any kind of arbitrage. Asset pricing models having underlying processes as martingale, rules out the existence of arbitrage, hence the study of martingale becomes vital. In this paper probability measure and conditional expectation are introduced to carry their intuitions in martingale process. Extensive programming efforts are incorporated through different *Mathematica* programs.

II. Literature survey

Binomial model for Stock prices:

For one period fixed fluctuation Binomial model, we consider $t = 0$ at starting and $t = 1$ for the end of period, S_0 as the stock price per share at $t = 0$ with $S_0 > 0$, S_1 as the stock price per share at $t = 1$, u as up factor of a stock price for one period, d as down factor of a stock price for one period, p as the probability of the increase of a stock price at next period, q as the probability of the decrease of a stock price at next period. Which leaves us with $p = 1 - q$.

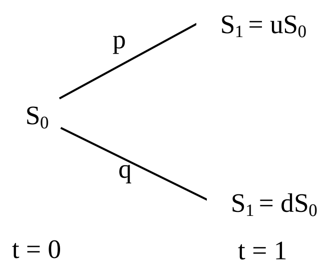

Fig. 1.1

Since the event of Stock price's increasing or decreasing is random, we are considering that a coin is tossed and the outcomes of the coin determine the price. Hence we get the following mathematical frame work for a Multi-period Binomial Model (in this case for three periods) where Head(H) as an output of the coin toss event means the increase of the stock price by the up factor u and Tell(T) means the decrease of the stock price by the down factor d.

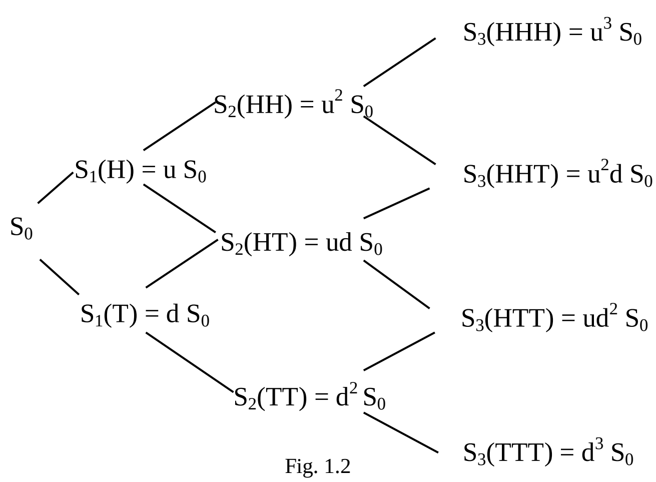

Fig. 1.2

Suppose, we begin with wealth X_0 and buy Δ_0 shares of stock at time zero. If $\Delta_0 S_0 \geq X_0$ then we have to borrow $(\Delta_0 S_0 - X_0)$ from the money market at interest rate r. Then the value of our portfolio of stock and money market at time one will be,

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$X_1 = \underline{\Delta_0 S_1} + \underline{(1 + r)(X_0 - \Delta_0 S_0)} = (1+r) X_0 + \Delta_0 [S_1 - (1+r) S_0]$

Risky part Risk free part

Here we can divide X_1 in to risky part $\Delta_0 S_1$ and risk free part $(1 + r)(X_0 - \Delta_0 S_0)$. Particularly,

$X_1(H) = \Delta_0 S_1(H) + (1 + r)(X_0 - \Delta_0 S_0)$

$X_1(T) = \Delta_0 S_1(T) + (1 + r)(X_0 - \Delta_0 S_0)$

Here we are not using any argument for X_0 and S_0 since X_0 and S_0 are not actually random. The randomness occurs only for X_1 and S_1. So without randomness the term, $(1 + r)(X_0 - \Delta_0 S_0)$ become risk less and the term $\Delta_0 S_1$ is the risky part since S_1 can be either $S_1(H)$ or $S_1(T)$. So clearly X_1 is another random variable.

We want to choose X_0 and Δ_0 in a way so that $X_1(H) = V_1(H)$ and $X_1(T) = V_1(T)$ where $V_1(H) = u\, S_0$ and $V_1(T) = d\, S_0$ are known as we have given the payoff function of the derivatives security for this one period model. Thus replicating the derivative security requires that

$$X_0 + \Delta_0 \left(\frac{1}{1+r} S_1(H) - S_0\right) = \frac{1}{1+r} V_1(H) \qquad (1.1)$$

$$X_0 + \Delta_0 \left(\frac{1}{1+r} S_1(T) - S_0\right) = \frac{1}{1+r} V_1(T) \qquad (1.2)$$

Now we will solve the equation for X_0 and Δ_0.

Multiplying (1.1) by $\tilde{p}$ and (1.2) by $\tilde{q} = 1 - \tilde{p}$ and adding them we get

$$X_0 + \Delta_0 \left(\frac{1}{1+r} [\tilde{p}\, S_1(H) + \tilde{q}\, S_1(T)] - S_0\right)$$

$$= \frac{1}{1+r} [\tilde{p}\, V_1(H) + \tilde{q}\, V_1(T)] \qquad (1.3)$$

If we choose $\tilde{p}$ in a way that

$$S_0 = \frac{1}{1+r} [\tilde{p}\, S_1(H) + \tilde{q}\, S_1(T)] \qquad (1.4)$$

then the term multiplying Δ_0 in (1.3) is zero and we have a simple formula for X_0

$$X_0 = \frac{1}{1+r} [\tilde{p}\, V_1(H) + \tilde{q}\, V_1(T)]$$

We can solve $\tilde{p}$ directly from (1.4) in the form

$$S_0 = \frac{1}{1+r}\left[\tilde{p} u S_0 + (1-\tilde{p}) d S_0\right]$$

$$= \frac{S_0}{1+r}\left[(u-d)\tilde{p} + d\right]$$

This leads to the formulas:

$$\tilde{p} = \frac{1+r-d}{u-d} \quad \text{and} \quad \tilde{q} = \frac{u-1-r}{u-d}$$

We call $\tilde{p}$ and $\tilde{q}$ as the *Risk neutral probabilities*.

Under the actual probabilities, the average rate of growth of the stock is typically strictly greater than the rate of growth of the same amount's investment in the money market. Otherwise no one would want to incur the risk associated with investing in money market. So for the actual probability p and $q = 1 - p$.

$\underline{p\, S_1(H) + q\, S_1(T)} > \underline{(1+r)\, S_0}$

in Stock market in Money market

$$\Rightarrow S_0 < \frac{1}{1+r} [p\, S_1(H) + q\, S_1(T)]$$

Whereas we chose $\tilde{p}$ and $\tilde{q} = 1 - \tilde{p}$ in such a way so that it satisfies

$$S_0 = \frac{1}{1+r} [\tilde{p}\, S_1(H) + \tilde{q}\, S_1(T)]$$

$$\Rightarrow (1+r) S_0 = \tilde{p}\, S_1(H) + \tilde{q}\, S_1(T)$$

Here the numbers $\tilde{p}$ and $\tilde{q}$ makes the average rate of growth of the stock exactly the same as the rate of growth of the money market investment. Then the investors must be neutral about risk (they do not require compensation for assuming it nor they are willing to pay extra for it.). Hence the name risk neutral probabilities.

$$\tilde{p} = \frac{1+r-d}{u-d} \text{ and } \tilde{q} = \frac{u-1-r}{u-d} \qquad (1.5)$$

We call the number $\tilde{p}$ and $\tilde{q}$ as risk neutral probabilities and the equation.

$$V_0 = \frac{1}{1+r} [\, \tilde{p}\, V_1(H) + \tilde{q}\, V_1(T)] \qquad (1.6)$$

as risk neutral pricing formula where the actual probabilities are absent.

For such use of risk neutral probabilities in risk neutral pricing formula the prices of the derivative security in the Binomial model depends only on the set of possible stock price paths, (pricing formula V_0 uses all V_1 ( ) where ( ) represents all possible stock price paths) but not on how probable these paths are. i.e. the probability of none of these paths appearing in the pricing formula.

Arbitrage free market:

Arbitrage is a trading strategy that begins with no money, has zero probability of losing money and has a positive probability of making money. By this trading strategy wealth can be generated from nothing. A market which is free from arbitrage is called arbitrage free market.

A mathematical model that admits arbitrage negatively influences the mathematical analysis. From mathematical point of view, an arbitrage free market must hold the following inequalities,

$0 < d < (1+ r) < u$

Where, r is the interest rate in the money market. Here we recall the one period fixed fluctuation model to explain these inequalities.

Note: $d > 0$ as the stock prices are always positive.

Explanation of d < (1 + r):

At first we assume the situation when $d \geq (1 + r)$. Now if one begin with zero wealth and at time t = 0 borrows S_0 from the money market in order to buy stock, then he has to pay the money market $(1 + r) S_0$ at time t = 1, while in the worst stock price fall he will get dS_0 which is greater than or equal to $(1 + r) S_0$, since $d \geq (1 + r) \Rightarrow dS_0 \geq (1 + r) S_0$.

And if the stock price increases he will get uS_0 which is strictly greater than $(1+r) S_0$, since $u > d \geq (1+r) \geq uS_0 > (1+r) S_0$, So for all sorts of fluctuation of stock prices he has the ability to pay off the money market debt and has a positive probability to gain wealth from nothing.

This provides an arbitrage. So we must have $d < (1+r)$ for arbitrage free market.

Explanation of u > (1+ r):

Similarly we consider the situation for $u \leq (1+r)$. Now if one sells the stock at time zero and invest the money in the money market, at the time one he will get $(1+r) S_0$ from the money market while in the best case, the cost of replacing the stock is uS_0 which is less than value of the money market return. Thus he has the stock at time t = 1, as he had at the time t = 0 plus he may have some profit from the money market, so this also provides an arbitrage. So we must have $u > (1+r)$ for arbitrage free market.

Expected values (mathematical expectations):

If X is a random variable defined over the sample space, $\Omega = \{\omega_1, \omega_2 \ldots\ldots\ldots\ldots \omega_n\}$ with corresponding probabilities $P(\omega_1), P(\omega_2) \ldots\ldots\ldots\ldots P(\omega_n)$ then the expected value, i.e. the mathematical expectation of X, is symbolically defined as

$$E(X) = \sum_{i=1}^{n} X(\omega_i) P(\omega_i) = \sum_{\omega \in \Omega} X(\omega)\, P(\omega)$$

Discounted process and Expected value:

Recalling from the multi period Binomial model and the equation (1.6) at every time n and for every sequence of coin tosses $\omega_1, \omega_2 \ldots\ldots \omega_n$, we have

$$S_n(\omega_1, \omega_2 \ldots\ldots \omega_n) =$$

$$\frac{1}{1+r} [\tilde{p}\, S_{n+1}(\omega_1, \omega_2, \ldots\ldots \omega_n, H) + \tilde{q}\, S_{n+1}(\omega_1, \omega_2 \ldots\ldots \omega_n, T)]$$

The stock price at time n is the discounted weighted average of the two possible stock prices at time t = n + 1, where the $\tilde{p}$ & $\tilde{q}$ are the weights used in the averaging. So the expected value of S_{n+1} at time t = n is,

$$E_n[S_{n+1}](\omega_1 \ldots\ldots \omega_n) =$$

$\tilde{p}\, S_{n+1}(\omega_1, \omega_2, \ldots \omega_n H) + \tilde{q}\, S_{n+1}(\omega_1, \omega_2 \ldots \omega_n T)$

From (2.1) & (2.2)we have

$$S_n = \frac{1}{1+r} \tilde{E}_n [S_{n+1}]$$

Conditional Expectation:

The conditional expectation of a stock price S_m at time t = n, where (n < m) is the expected values of all possible values of S_m at time t = n, conditioned by their information we need at time t = n. We denote it by $E_n[S_m]$.

In general let n satisfy $1 \le n \le N$ and let $\omega_1, \omega_2, \ldots, \omega_n$ be given and for the moment fixed. There are 2^{N-n} possible continuations $\omega_{n+1}, \omega_{n+2}, \ldots, \omega_N$ of the fixed sequence $\omega_1, \omega_2, \ldots, \omega_n$ denoted by $\# H(\omega_{n+1}, \omega_{n+2}, \ldots, \omega_N)$ the number of heads in the continuation $\omega_{n+1}, \omega_{n+2}, \ldots, \omega_N$ and denoted by $\# T(\omega_{n+1}, \omega_{n+2}, \ldots, \omega_N)$ the number of tails.

$E_n[X](\omega_1, \omega_2, \ldots, \omega_N) =$

$$\sum_{\omega_{n+1},\ldots,\omega_N} p^{\# H(\omega_{n+1},\ldots\omega_N)} q^{\# T(\omega_{n+1},\ldots\omega_N)} X(\omega_1,\ldots\omega_n\omega_{n+1}\ldots\omega_N) \quad (1.7)$$

Based on what we know at time zero, the conditional expectation $E_n[X]$ is random in the sense that its value depends on the first n coin tosses, which we do not know until time n.

The two extreme cases of conditioning are $E_0[X]$, the conditional expectation of X based on no information, which we define by, $E_0[X] = EX$, where EX means the total expectation using complete continuations $\omega_1, \omega_2, \ldots, \omega_n, \omega_{n+1}, \omega_{n+2}, \ldots, \omega_N$ and $E_N[X]$ the conditional expectation of X based on knowledge of all n coin tosses, which are defined by $E_n[X] = X$, which is obtained by no continuations.

Martingale:

In probability theory, a martingale is a stochastic process (i.e. a sequence of random variables) such that the conditional expected value of an observation at some time t, given all the observations up to some earlier time s, is equal to the observation at that earlier time s.

Consider the Binomial asset pricing model. Let $M_0, M_1, M_2, \ldots M_N$ be a sequence of random variables, with each M_N depending on random evolution up to times n (i.e. M_0 constant). Such a sequence of random variables is called a Martingale stochastic process,

if $M_n = E_n[M_{n+1}]$, n = 0, 1, 2, …N-1

i.e. for martingale process the conditional expected value of the next observation, given all the past observations, is equal to the last observation.

Sub-martingale:

The above process $(M_0, M_1, M_2, \ldots M_N)$ is called as sub martingale if $M_n \le E_n[M_{n+1}]$, for n = 0,1, 2…...N-1

Super-martingale:

The above process $(M_0, M_1, M_2, \ldots M_N)$ is called as super martingale if $M_n \ge E_n[M_{n+1}]$, for n = 0 ,1, 2…...N-1

First fundamental theorem of Asset pricing:

"If we can find a risk- neutral measure in a model (i.e. a measure that agrees with the actual probability measure about which paths have zero probability and under which the discounted prices of all primary assets are martingale), then there is no arbitrage in the model."

The main importance of this theorem is in ruling out the arbitrage from the market.

III. OBJECTIVE OF THE STUDY

The main objective of this study is to design a structure based on the first fundamental theorem of Asset pricing to identify the existence of any arbitrage in a stock market. Here we are trying find a nice graphical representation using necessary known inputs like initial stock price, up factor and down factor (which can be predicted from the underlying asset's history), So that the existence of arbitrage is easily detectable.

IV. METHODOLOGY

A program in *Mathematica* generated by using the last formula stated may be used to determine conditional expectation for any time period. The conditional expectation of S_6 based on the information available at time t = 3, was determined by the program as,

$$E_3[S_6](HHH) = E_3[S_6](66.55) = 79.2621$$

$$E_3[S_6](HHT) = E_3[S_6](58.685) = 69.8948$$

$$E_3[S_6](THT) = E_3[S_6](51.7495) = 61.6345$$

$$E_3[S_6](TTT) = E_3[S_6](45.6337) = 54.3504$$

where the inputs were, initial stock price S_0 = 50, up factor u = 1.1, down factor d = 0.97, interest rate r = 0.06.

If we plot M_n along horizontal axes (X axes) and $E_n[M_{n+1}]$ along vertical axes (Y axes) i.e. if (M_n , $E_n[M_{n+1}]$) represents a point in that rectangular plane, then when the process M_n is martingale X co-ordinate and Y co-ordinate of every points are equal and so every point will lie on the diagonal line. If the process is super martingale then the points will lie above the diagonal line and for sub martingale they will lie below the diagonal.

V. ANALYSIS AND FINDINGS

First let us verify for discounted stock price and risk neutral probability. When the program runs the input box will appear with the tags as follows.

"Input the number of last period"

We enter 8. (but this program will work for any number of periods)

Then it asks for the types of the input. In this case if we want to use the input as 1 then the program will ask for the specific values of stock price, up & down factor, interest. Otherwise we use the input is 0 (zero) to use the default values as follows,

Stock price S_0 = 50

Up factor u = 1.3

Down factor d = 0.8

Interest rate (money market) r = 0.12.

The next option is to decide about to check for the Stock price (input 1) or Discounted stock price (input 2). We choose the 2nd option.

The last option is to decide about choice of kind of probability. We can use either risk neutral probability (input 1) or random probability (input 2).

Here the program generates probability randomly using the command "Random[ ]" of *Mathematica* that returns random number between 0 (zero) and 1. Finally we get the following output.

$$E_7[S_8](HHHHHHH) = E_7[S_8](141.921) = 141.921$$

$$E_7[S_8](HHHHHHT) = E_7[S_8](87.3361) = 87.3361$$

$$E_7[S_8](THHHHHT) = E_7[S_8](53.7453) = 53.7453$$

$$E_7[S_8](TTHHHHT) = E_7[S_8](33.074) = 33.074$$

$$E_7[S_8](TTTHHHT) = E_7[S_8](20.3533) = 20.3533$$

$$E_7[S_8](TTTTHHT) = E_7[S_8](12.5251) = 12.5251$$

$$E_7[S_8](TTTTTHT) = E_7[S_8](7.70774) = 7.70774$$

$$E_7[S_8](TTTTTTT) = E_7[S_8](4.74323) = 4.74323$$

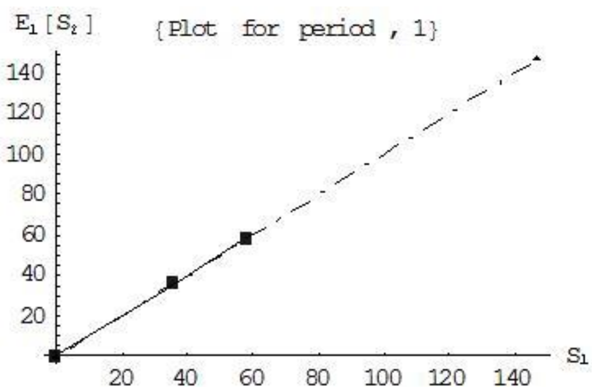

Fig. 5.1

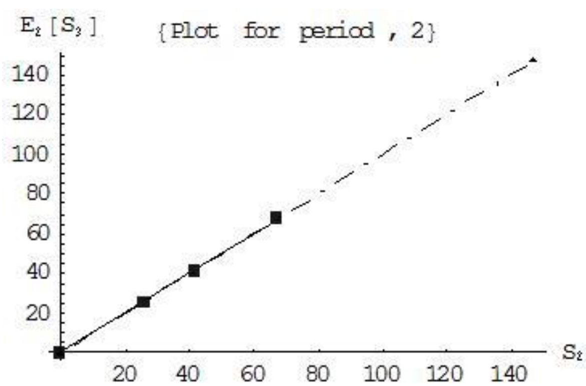

Fig. 5.2

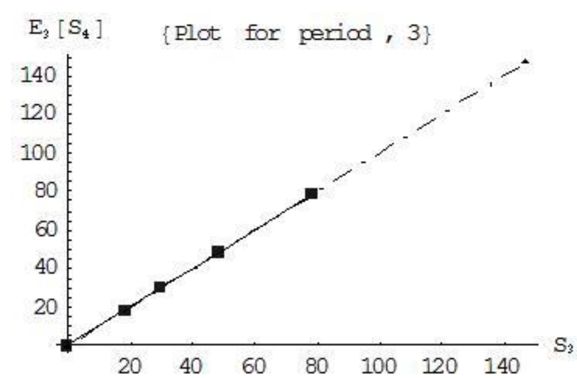

Fig 5.3

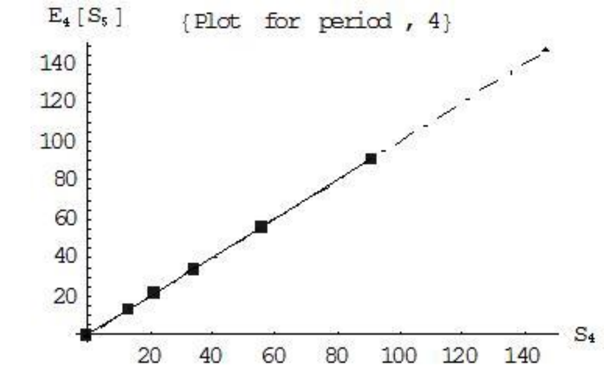

Fig. 5.4

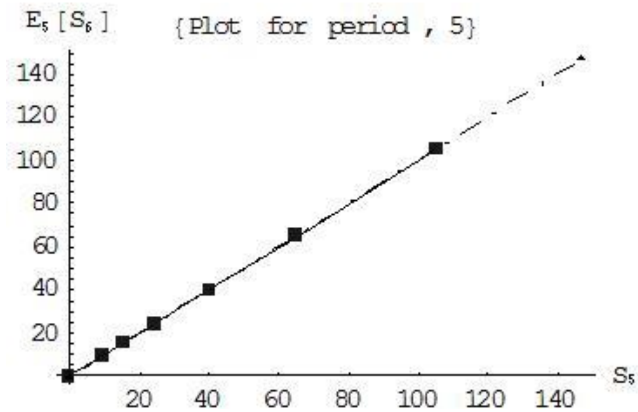

Fig. 5.5

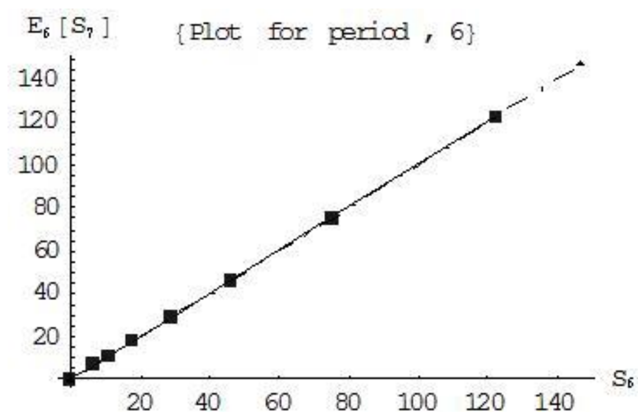

Figure 5.6

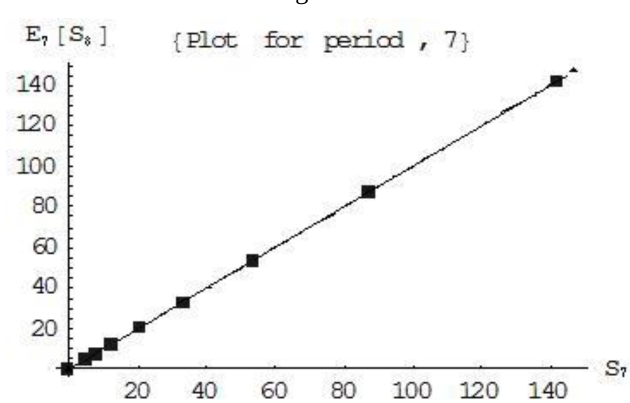

Fig. 5.7

Since every point is lying on the diagonal line the process is a Martingale process.

Now to test for risk neutral probability and stock price without discounting we input 8,0,1,1 consecutively in the input box when it appears.

$$E_7[S_8](HHHHHHH) = E_7[S_8](313.743) = 351.392$$

$$E_7[S_8](HHHHHHT) = E_7[S_8](193.072) = 216.241$$

$$E_7[S_8](THHHHHT) = E_7[S_8](118.814) = 133.071$$

$$E_7[S_8](TTHHHHT) = E_7[S_8](73.1162) = 81.8901$$

$$E_7[S_8](TTTHHHT) = E_7[S_8](44.9946) = 50.3939$$

$$E_7[S_8](TTTTHHT) = E_7[S_8](27.698) = 31.0116$$

$$E_7[S_8](TTTTTHT) = E_7[S_8](17.0394) = 19.0841$$

$$E_7[S_8](TTTTTTT) = E_7[S_8](10.4858) = 11.7441$$

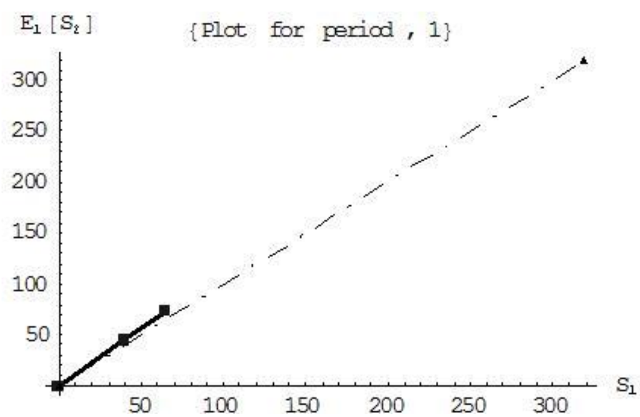

Fig. 5.8

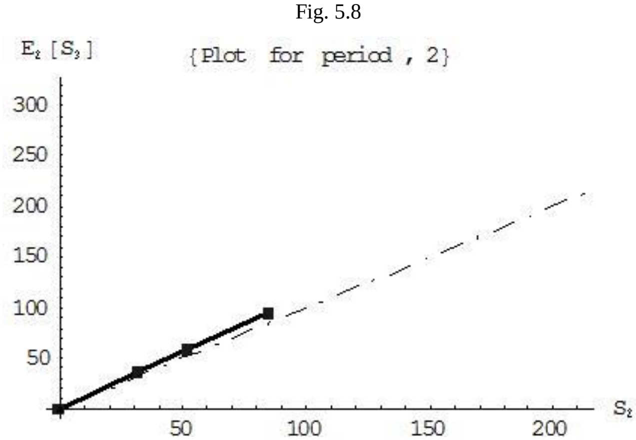

Fig. 5.9

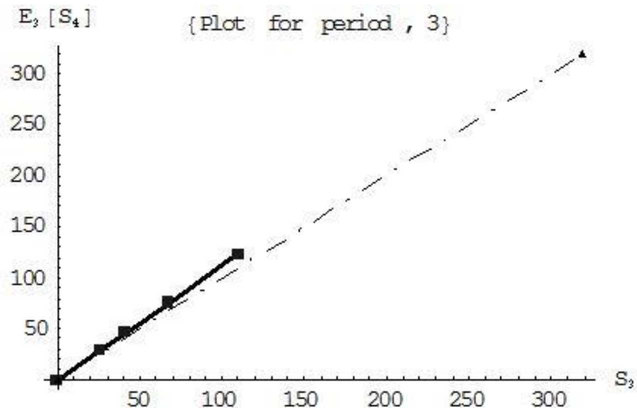

Fig. 5.10

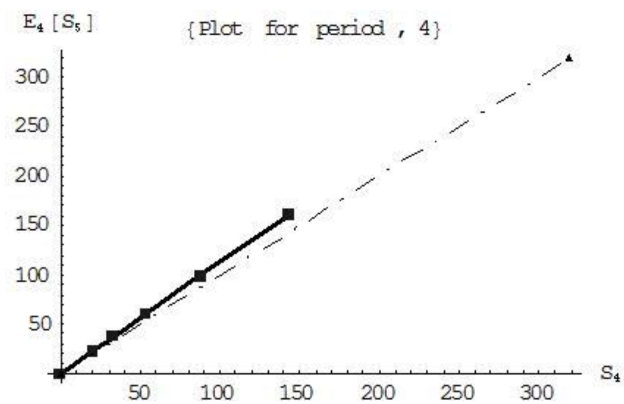

Fig. 5.11

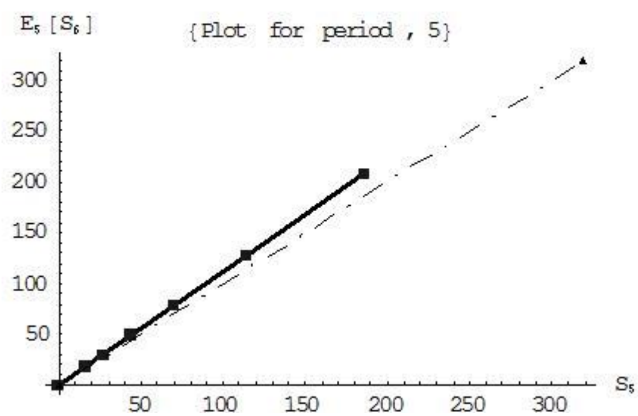

Fig. 5.12

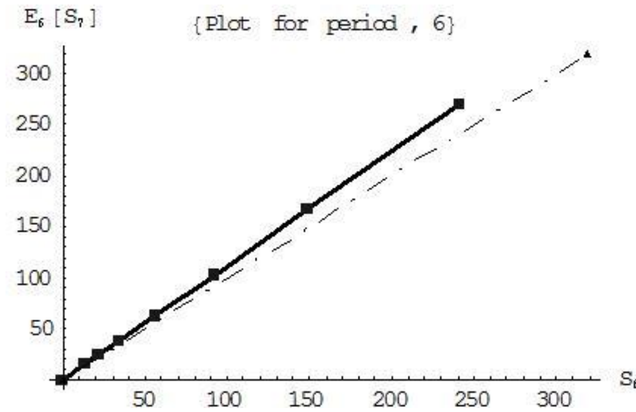

Fig. 5.13

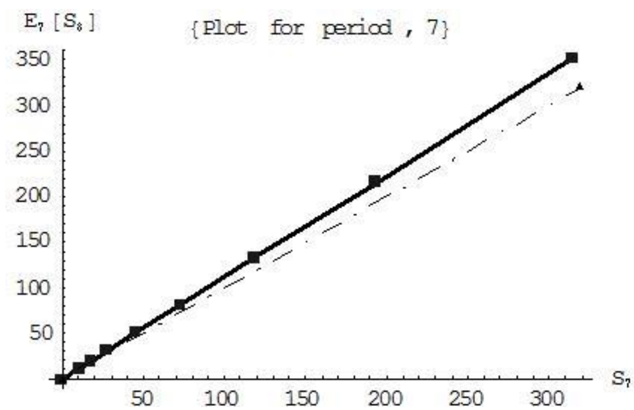

Fig. 5.14

Since every point is lying over the diagonal line the process is a super-martingale process.

For stock price process under random probability we input consequently 8,0,1,2 where the random probability p > p` where p` is the solution of the equations[1],

$$p` \times u + q` \times d = 1,\ p` + q` = 1 \quad (5.1)$$

to get the following output

$$E_7[S_8](HHHHHHH) = E_7[S_8](313.743) = 273.508$$

$$E_7[S_8](HHHHHHT) = E_7[S_8](193.072) = 168.313$$

$$E_7[S_8](THHHHHT) = E_7[S_8](118.814) = 103.577$$

$$E_7[S_8](TTHHHHT) = E_7[S_8](73.1162) = 63.7396$$

$$E_7[S_8](TTTHHHT) = E_7[S_8](44.9946) = 39.2244$$

$$E_7[S_8](TTTTHHT) = E_7[S_8](27.698) = 24.1381$$

$$E_7[S_8](TTTTTHT) = E_7[S_8](17.0394) = 14.8542$$

$$E_7[S_8](TTTTTTT) = E_7[S_8](10.4858) = 9.14105$$

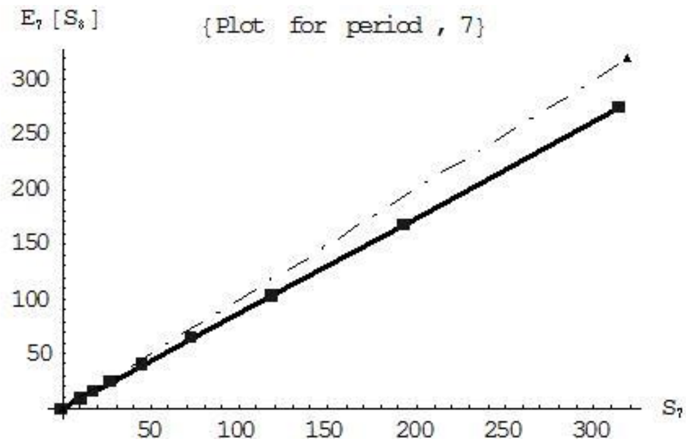

Fig. 5.15

Which is the graph for the last period only. Since every point is lying under the diagonal line this process is a sub-martingale process.

VI. DISCUSSIONS

It has been observed that if we run the program frequently for the series of inputs as 8,0,1,2 using random probability instead of risk neutral probability, sometimes it shows stock price process is super martingale and sometimes sub martingale. So there exists a probability measure for which stock price process is martingale. From the theoretical point of view[1] the measure is obtained by solving the equations 5.1.

It is observed that when the value of p assigned by the random number generator "Random[ ]" = p > p` the stock price process is super martingale and when p < p` the stock price process is sub martingale. Here for the default values of the parameters u = 1.3 and d = 0.8, the value of p` is 0.4 which is obtained by solving the equations 5.1 and p = 0.143517 assigned by the random number generator. As a result in the last graph it shows that the stock price process is sub martingale.

So "p(random)<p`" leads the stock price process to sub martingale and "p(random)>p`" leads the stock price process to super martingale. And for a variable to be martingale the S_{n+1} VS $E_n[S_{n+1}]$ graph must coincide with the diagonal line for every value of n.

But if we consider the stock price which is super martingale for the given set of data, it is observed that with the increase of n (number of period) the slope of the straight line increases i.e. the ratio of expectation of the stock price at next period (n+1) with the stock price at n is increasing with the increase of stock price. So the quantity of gained money by an arbitrager will increase with the increase of stock price i.e. He or she will gain as more as the stock price when the stock price is super martingale. The same reason will explain the loss of an arbitrager in case of sub martingale.

VII. CONCLUSION

Confinement to the discrete settings is motivated by the fact that grasping the intuitions is convenient in discrete settings. The intuitions like risk-neutral probabilities, conditional expectations, martingale and different stochastic processes (involved in discrete time binomial asset pricing model) as explored and programmed paves the path of assuming further research in many of the challenging directions of Mathematical Finance. The intuition of measurability is also important for the functions of random variable and for the stochastic process in this context. Other asset pricing models like Black-Scholes model, Cox-Ingersoll-Ross model (CIR model) etc. which are still in use in the market with some changes can be used to explore similar ideas through the method used in this paper.

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