Dynamics of Modified Leslie-Gower Predator-Prey Model with Predator Harvesting

K. Saleh
Department of Mathematics, King Fahd University of Petroleum and Minerals, Dhahran, 31261 Saudi Arabia
E-mail: khairul@kfupm.edu.sa

***Abstract*-- In this paper we discuss dynamical properties of a modified Leslie-Gower predator-prey system with quadratic predator harvesting. We show that all solutions of the system are uniformly bounded. The permanence, stability (of equilibrium points) and bifurcations of the system are investigated. The system has at most two equilibria in the interior of the first quadrant and can exhibit Bogdanov–Takens, Hopf, transcritical and saddle-node bifurcations. Depending on the values of parameters, the system may have a stable periodic solution, or a homoclinic loop, or a saddle point, or a stable focus. Biologically, both populations prey and their predators may survive for certain parameter values.**

***Index Term*— Modified Leslie-Gower, predator-prey, quadratic harvesting, bifurcations, equilibria.**

I. Introduction

Let $x(t)$ and $y(t)$ denote densities of prey and predators at time t respectively. Consider the modified Leslie-Gower predator-prey model [6]

$$\begin{aligned} \dot{x} &= rx\left(1-\frac{x}{K}\right)-\frac{c_1\,xy}{m_1+x} \\ \dot{y} &= sy\left(1-\frac{c_2\,y}{m_2+x}\right), \end{aligned} \tag{1}$$

where c_1, c_2, K, m_1, m_2, r and s are positive. The coefficients r and K represent the intrinsic growth rate and environmental carrying capacity for the prey in the absence of predation, respectively. The natural growth rate of the predator is given by s, while sc_2 is the maximum value of the per capita reduction rate of predators. The maximum value of the per capita reduction rate of prey species is denoted by c_2, while m_1 and m_2 measure the extent to which the environment provides protection to prey and predator, respectively.

System (1) has been investigated by several researchers. In particular, the boundedness of solutions and global stability of the positive equilibrium points of the system has been studied by Aziz-Alaoui and Daher Okiye [6]. Sufficient conditions for the existence and global attractivity of positive periodic solutions of the model were discussed by Zhu and Wang [20]. Gupta and Chandra [10] studied bifurcations of model (1), taking into account Michaelis-Menten type prey harvesting. They analyzed the effect of prey harvesting and growth rate of predator on the model as these terms are important from the ecological point of view. It was shown that the model displays a complex dynamics in the prey-predator plane. The permanence, stability and bifurcations of the model were discussed.

In this paper, we study model (1) with quadratic predator harvesting. Taking into account the predator harvesting and applying a linear scaling on x, y, t and all parameters in system (1), we obtain the following system of differential equations:

$$\begin{aligned} \dot{x} &= x\left(1-x-\frac{\alpha y}{m+x}\right) \\ \dot{y} &= \rho y\left(1-\frac{\beta y}{m+x}-\delta y\right), \end{aligned} \tag{2}$$

with initial conditions

$$x(0)=x_0>0, \qquad y(0)=y_0>0. \tag{3}$$

The coefficients $\alpha, \beta, \delta, \rho$ and m depend on parameters in system (1). Here $\rho\delta$ represents the rate of harvesting of the predator. In [7, 9, 15, 16, 17] the term $-\rho\delta y^2$ is considered as competition amongst predators. There the authors investigated dynamics of predator-prey models with different types of response functions. For more details on the linear scaling, see [10].

The investigation concerns the dynamics of system (2) in the closed first quadrant $\mathbb{R}^2_+$. We shall show that system (2) possesses at most two equilibria in the interior of the first quadrant, and three trivial equilibria coexist on axes for all parameter values. Furthermore, we show that the system can exhibit many types of bifurcation phenomena, including saddle-node, transcritical, hopf and the codimension two Bogdanov-Takens bifurcations.

This paper is organized as follows. In Section 2, we investigate the existence and stability of equilibria. Dynamical behavior in the small neighborhood of the equilibria is also discussed. We show that all solutions of system (2) are trapped in a finite domain in the first quadrant. Several phase portraits of system (2) without periodic orbit are depicted. Section 3 discusses bifurcations occurring in the system, depending on all parameter values. We show that the system possesses saddle-nodes, subcritical Hopf, transcritical and Bogdanov-Takens bifurcations. The paper is ended with a brief discussion on the current and future researches.

II. Boundedness and Permanence

As mentioned in the previous section, we are interested only in the dynamics of system (2) in the closed first quadrant $\mathbb{R}_+^2$. It is easy to see that the positive x-axis and the positive y-axis are both invariant under the flow. This implies that all orbits that start in $\mathbb{R}_+^2$ will not leave $\mathbb{R}_+^2$. In other words; the closed first quadrant $\mathbb{R}_+^2$ is positively invariant under the flow generated by system (2).

A. *Boundedness of Solutions*

Now we prove that all solutions of system (2) with initial conditions (3) are trapped in a finite region subset $\mathbb{R}_+^2$. We use the following two lemmas (see [10, 21]) to prove the boundedness and permanence of system (2).

Lemma 2.1 If $a, b > 0$ and $\frac{dP}{dt} \leq (\geq)\, P(a - bP)$ with $P(0) > 0$, then

$$\limsup_{t\to+\infty} P(t) \leq \frac{a}{b}\left(\liminf_{t\to+\infty} P(t) \geq \frac{a}{b}\right)$$

Observe that Lemma 2.1 is quantitatively equivalent to the following lemma [10].

Lemma 2.2 If $a, b > 0$ and $\frac{dP}{dt} \leq (\geq)\, P(a - bP)$ with $P(0) > 0$, then for all $t \geq 0$,

$$P(t) \leq \frac{a}{b - Ce^{at}}, \quad C = \frac{bP(0) - a}{P(0)}.$$

In particular, $P(t) \leq \max\left\{P(0), \frac{a}{b}\right\}$ for all $t \geq 0$.

Theorem 2.1 All solutions of system (2) with initial conditions (3) are uniformly bounded.

Proof. Since x and y are both positive, from (2) we can write

$$\dot{x} = x\left(1 - x - \frac{\alpha y}{m + x}\right) \leq x(1 - x) \tag{4}$$

From Lemma 2.2, we have

$$x(t) \leq max\{x(0), 1\} \equiv N_1 \text{ for all } t \geq 0.$$

From the predator equation of system (2) we have

$$\dot{y} = \rho\, y\left(1 - \frac{\beta\, y}{m + x} - \delta y\right) \leq \rho y\left[1 - \left(\delta + \frac{\beta}{m + N_1}\right) y\right]. \tag{5}$$

Lemma 2.2 implies that

$$y(t) \leq \max\left\{ y(0), \frac{m + N_1}{\delta(m + N_1) + \beta}\right\} \equiv N\_2$$

∎

B. *Permanence*

In what follows, we prove permanence of system (2) under a certain condition. First, we recall the definition of permanence, see [22].

Definition 2.1 System (2) with initial conditions (3) is permanent if there are positive constants C_1 and C_2 $(0 < C_1 < C_2)$ such that each positive solution $(x(t), y(t))$ of the system satisfies:

$$\min\left\{\liminf_{t\to+\infty} x(t),\ \liminf_{t\to+\infty} y(t)\right\} \geq C_1$$
$$\max\left\{\limsup_{t\to+\infty} x(t),\ \limsup_{t\to+\infty} y(t)\right\} \leq C_2$$

Theorem 2.2 System (2) with initial conditions (3) is permanent if

$$\alpha < \delta\, m + \frac{\beta\, m}{m + 1}.$$

Proof. Observe that for sufficiently large value of t,

$$0 \leq x(t) \leq 1 \ \ and \ \ y(t) \leq \frac{m + 1}{\delta(m + 1) + \beta}.$$

Thus, from the prey equation of (2), we have

$$\dot{x} = x\left(1 - x - \frac{\alpha\, y}{m + x}\right) \geq x(\varphi_1 - x),$$

where

$$\varphi_1 = 1 - \frac{\alpha(m + 1)}{\delta\, m(m + 1) + \beta m}.$$

From the predator equation of (2), we can write

$$\dot{y} = \rho\, y\left(1 - \frac{\beta\, y}{m + x} - \delta\, y\right) \geq \rho\, y\left(1 - \left(\delta + \frac{\beta}{m}\right) y\right).$$

Lemma 2.1 implies that

$$\liminf_{t\to+\infty} y(t) \leq \frac{m}{\beta + \delta m} \equiv \varphi_2.$$

Form (4) and (5), together with Lemma 2.1, we obtain

$$\limsup_{t\to+\infty} x(t) < 1 \text{ and } \limsup_{t\to+\infty} y(t) < \frac{m + N_1}{\delta(m + N_1) + \beta}.$$

Putting

$$C_1 = min\{\varphi_1, \varphi_2\} \text{ and } C_2 = max\left\{1, \frac{m + N_1}{\delta(m + N_1) + \beta}\right\}$$

implies the permanence of system (2) ■

III. EXISTENCE AND STABILITY OF SOLUTIONS

The general Jacobian matrix of system (2) is given by

$$J(x, y) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad (6)$$

where

$$a_{11} = 1 - 2x + \frac{\alpha\, y}{m + x}\left[\frac{x}{m + x} - 1\right], \qquad a_{12} = -\frac{\alpha\, x}{m + x},$$

$$a_{21} = \frac{\beta\delta\, y^2}{(m + x)^2}, \qquad a_{11} = \rho\left(1 - 2y\left[\delta + \frac{\beta}{m + x}\right]\right).$$

A. Trivial Equilibria

System (2) possesses three trivial equilibrium points that appear for any choice of positive values of parameters:

1) the origin O(0,0);
2) the predator extinction point $P_1(1,0)$;
3) the prey extinction point $P_2\left(0, \frac{m}{\beta + \delta m}\right)$.

Theorem 3.1 For system (2),

a) the origin is always unstable;
b) the equilibrium P_1 is always a saddle point;
c) the equilibrium P_2 is stable if $\alpha > \beta + m\delta$ and a saddle point if $\alpha < \beta + m\delta$.

Proof.

a) The Jacobian matrix of the system at the origin is given by

$$J_0 = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}.$$

Eigenvalues of J_0 are $\lambda_1 = 1$ and $\lambda_2 = \rho > 0$. Thus, the origin is unstable.

b) At P_1, the Jacobian matrix is

$$J_1 = \begin{pmatrix} -1 & -\frac{\alpha}{1 + m} \\ 0 & \rho \end{pmatrix}.$$

Eigenvalues of J_1 are -1 and $\rho > 0$. We conclude that P_1 is a saddle-point for any choice of positive values of parameters.

c) At P_2, the Jacobian matrix is given by

$$J_2 = \begin{pmatrix} 1 - \frac{\alpha}{\beta + m\delta} & 0 \\ \frac{\beta\rho}{(\beta + m\delta)^2} & -\rho \end{pmatrix}.$$

Eigenvalues of J_2 are $1 - \frac{\alpha}{\beta + m\delta}$ and $-\rho < 0$. Hence the result follows. ■

B. Positive Equilibria

Two possible interior equilibrium points of system (2) are $E_1(x_1, y_1)$ and $E_2(x_2, y_2)$, where for $i = 1,2$,

$$x_i = \frac{\delta(1 - m) - \beta + (-1)^{i+1}\sqrt{\Delta}}{2\delta}, \quad y_i = \frac{(1 - x_i)(m + x_i)}{\alpha},$$

$$\Delta = (\beta + \delta - m\delta)^2 + 4\delta(\beta + m\delta - \alpha).$$

The following simple lemma gives the number of equilibria of system (2) in the interior of the first quadrant. The proof is straightforward and is being omitted.

Lemma 3.1

i. System (2) has no equilibrium in the interior of the first quadrant if $\Delta < 0$.
ii. System (2) has two equilibria in the interior of the first quadrant, which are E_1 and E_2 if for $i = 1,2$,

$$\Delta > 0, \quad 0 < x_1 < 1.$$

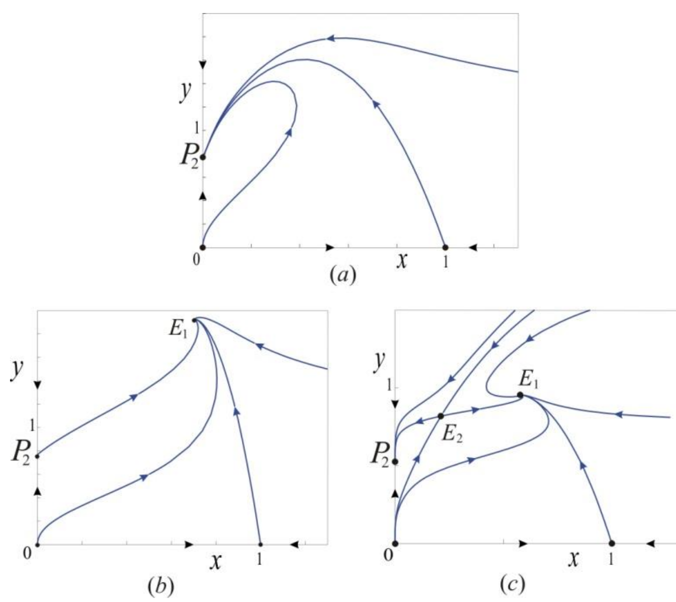

Fig. 1. Phase portraits of system (2) without periodic orbits: (a) no equilibrium in the interior of the first quadrant. All orbits are attracted to equilibrium P_2; (b) one stable focus point appears in the interior of the first quadrant; (c) two equilibria coexist in the interior of the first quadrant: one stable focus and one saddle point.

Figure 1 provides several phase portraits of system (2). The trajectories are depicted using a MATLAB package called Matcont [13]. The cases where system (2) has zero, one and two equilibrium points in the interior of the first quadrant occur. In Figure 1(a), the system has no equilibrium in the interior of $\mathbb{R}_+^2$. All trajectories tend to the prey extinction equilibrium P_2. This means that independent on the initial numbers of the interacting populations, only predators survive. Figure 1(b) gives a phase portrait with one equilibrium point in the interior of the first quadrant. The point E_1 is a stable focus. All trajectories go to this focus point when $t \to \infty$. This means that both prey and predator populations survive for all positive initial values. The equilibrium P_2 is a saddle point with y-axis as its stable manifold. In Figure 1(c), we see two equilibria E_1 and E_2 coexist. The system has two stable equilibria P_2 and E_1. This means that depending on the initial conditions, either only the predators survive or both populations survive.

In the following theorem, we give conditions in which the equilibrium P_2 is a global attractor; see Figure 1(a) as an illustration of this phenomenon.

Theorem 3.2 If $\Delta < 0$ and $\alpha > \beta + m\delta$, then the equilibrium point P_2 is a global attractor in $\mathbb{R}_+^2$.

Proof. Observe that there is no equilibrium in the interior of the first quadrant when $\Delta < 0$. From Theorem 3.1; we know that the origin is a source, and P_1 is a saddle point. Moreover, the prey extinction equilibrium P_2 is a sink when $\alpha > \beta + m\delta$. Hence, the conclusion of the theorem holds. ■

IV. Bifurcations of System (2)

In this section, we investigate several bifurcations that occur in system (2), including saddle-node, transcritical, Hopf and Bogdanov-Takens bifurcations.

A. *Saddle-Node Bifurcations*

From the expressions of equilibria E_1 and E_2 we know that there is a region in the parameter space $(\alpha, \beta, \delta, m, \rho)$,

$$SN = \left\{ (\alpha, \beta, \delta, m, \rho) \mid \alpha = \beta + \delta\, m - \frac{(\beta + (m-1)\delta)^2}{4\delta} \right\},$$

such that for all points on the region SN, equilibria E_1 and E_2 collide and system (2) has a unique saddle-node equilibrium point $E_1 = E_2 = (x^*, y^*)$ where

$$x^* = \frac{\delta(1-m) - \beta}{2\delta}, \qquad y^* = \frac{(1-x^*)(m+x^*)}{\alpha}.$$

These two equilibria disappear when parameter values $(\alpha, \beta, \delta, m, \rho)$ pass from one side of the surface to the other side. This implies that system (2) possesses a saddle-node bifurcation of codimension one with SN as the saddle-node bifurcation surface.

B. *Transcritical Bifurcations*

System (3) undergoes a transcritical bifurcation involving the two equilibrium solutions E_1 and P_2. When $\alpha > \beta + m\delta$, the equilibrium E_1 is a stable focus or node, while P_2 is a saddle point. When $\alpha = \beta + m\delta$, equilibria E_1 and P_2 coincide and become $E_1 = P_2 = \left(0, \frac{m}{\alpha}\right)$. When $\alpha < \beta + m\delta$, both equilibria exchange their stabilities as E_1 becomes a saddle and leaves the first quadrant while P_2 becomes a stable focus or node. Figure 2 illustrates the transcritical bifurcation in the xy-plane. A scenario in which an equilibrium point enters or leaves the first quadrant via the prey extinction equilibrium P_2.

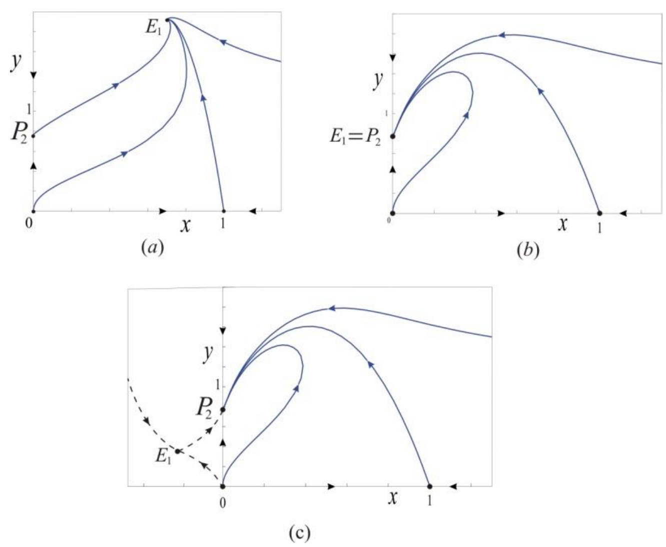

Fig. 2. A scenario of codimension one transcritical bifurcation: (a) a stable focus E_1 exists in the interior of the first quadrant; (b) equilibria E_1 and P_2 collide; (c) equilibrium E_1 leaves the first quadrant and become a saddle point.

Theorem 4.1 If $\alpha = \beta + m\delta$, then

a) *equilibrium points E_1 and P_2 collide, and*

b) *system (2) has one equilibrium point in the interior of the first quadrant when $\delta > \alpha$ and $m + 2(\delta - \alpha)(1-m) > 4(\delta - \alpha)$.*

Proof. Let $\alpha = \beta + m\delta$. From (7), we obtain a)

$$E_1 = (x_1, y_1) = \left(\delta(1-m) - \beta + \beta + (m-1)\delta, \frac{m}{\alpha}\right) = \left(0, \frac{m}{\alpha}\right) = P_2.$$

Again from (7), we have

$$E_2 = (x_2, y_2) = \left(2(\delta-\alpha), \frac{m + 2(\delta-\alpha)(1-m) - 4(\delta-\alpha)^2}{\alpha}\right).$$

It is clear that E_2 is in the interior of the first quadrant if two inequalities in b) hold. ■

C. *Hopf Bifurcations*

In Figure 1 we see that equilibrium E_1 is a focus point. At E_1, eigenvalues of the Jacobian matrix $J(x_1, y_1)$ are complex conjugates. Thus, it is possible that by changing parameter values, these eigenvalues are purely imaginary. In this case, the equilibrium point is a center-type non-hyperbolic equilibrium. Numerically, we find some parameter values at which the point has purely imaginary eigenvalues. Thus, system (2) may undergo Hopf bifurcation. The trace of the Jacobian matrix of system (2) at E_1 is given by

$$Tr\,[J(x_1, y_1)] = 1 - 2x + \frac{\alpha\, y}{m+x}\left(\frac{x}{m+x} - 1\right) + \rho\left[1 - 2y\left(\delta + \frac{\beta}{m+x}\right)\right]$$

The case when

$$Tr(J) = 0 \text{ and } \Omega \equiv a_{11}^2 + 4\,a_{12}\,a_{21} - 2a_{11}a_{22} + a_{22}^2 < 0$$

corresponds to the Andronov-Hopf bifurcation at which the equilibrium point changes its stability with appearance/disappearance of a periodic orbit. Let

$$H = \{(\alpha, \beta, \delta, m, \rho) | Tr(J) = 0, \quad \Omega < 0\}.$$

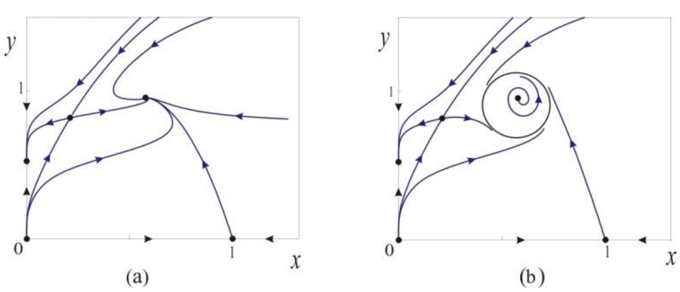

Fig. 3. A scenario of codimension one subcritical Hopf bifurcation of system (2): (a) a stable focus E_1 exists in the interior of the first quadrant; (b) a stable periodic orbit appears when parameter values cross the region H. The stable focus becomes unstable.

The set H is a Hopf bifurcation manifold of system (2). The stability of the periodic orbit created by the Hopf bifurcation is determined by the first Lyapunov value σ, see [4]. By calculating σ using Mathematica, we obtain that σ is negative at some points of H (we do not write the expression of σ in this paper, since it is too long). This means that system (2) undergoes subcritical Hopf bifurcations. As an implication, we have obtained a proof of the following proposition. Figure 3 illustrates a scenario of the subcritical Hopf bifurcation in system (2).

Proposition 4.1

a) If the parameter $(\alpha, \beta, \delta, m, \rho)$ is in H, then the equilibrium E_1 is a stable weak focus.
b) There exist some parameter values such that system (2) has at least one stable periodic orbit.

D. *Bogdanov-Takens Bifurcations*

The linear condition for Bogdanov-Takens bifurcation is that the Jacobian matrix (6) has double-zero eigenvalues with nontrivial nilpotent part. At a Bogdanov-Takens bifurcation point, a Hopf curve and a saddle-node curve are tangent to each other. There is also a homoclinic bifurcation occurring at another tangent curve, compare [17].

On the set

$$BT = \left\{(\alpha, \beta, \delta, m, \rho) \middle| \alpha = \beta + \delta\, m - \frac{(\beta + (1-m)\delta)^2}{4\delta}, \Omega = 0\right\}$$

the Jacobian matrix has double-zero eigenvalues with nontrivial nilpotent part. Thus, the system undergoes Bogdanov-Takens bifurcations on BT. As a consequence, we have the following.

Proposition 4.2 There exist values of the parameters such that system (2) has a unique unstable limit cycle for some parameter values, and system (2) has an unstable homoclinic loop for other parameter values.

V. DISCUSSION

In this paper, we have extended previous research on the modified Leslie-Gower predator-prey system by imposing quadratic harvesting components in the predator equation. We have shown that system (2) can have one stable equilibrium point in the interior of the first quadrant. This means that coexistence of both populations is possible under the quadratic predator harvesting policy. We also show that all solutions are trapped in a finite domain in the interior of the first quadrant and that the system exhibits interesting dynamics around the coexistence equilibria, including multiple bifurcations, periodic orbits and homoclinic orbits.

Biologically, there is still a lot of work to do in this direction. For example, it would be an idea to study the behavior of model (2) when harvesting is applied to both prey and predator equations, since people usually would like to harvest both populations. We may also consider other different avenues for future research in predator-prey modeling, including an analysis of modified Leslie-Gower predator-prey models with seasonal harvesting in one or both populations. In the latter case, the system may undergo chaotic behavior where we have strange attractors. The analysis performed and techniques used in this paper can be applied to problems other than predator-prey, especially two-dimensional ordinary differential equations.

REFERENCES

[1] H.R. Akcakaya, R. Arditi, L.R. Ginzburg, Ratio-dependent prediction: An abstraction that works, *Ecology* **79**L (1995) 995–1004.

[2] P.A. Abrams, L.R. Ginzburg, The nature of predation: Prey dependent, ratio-dependent or neither? *Trends in Ecology and Evolution* **15** (2000) 337–341.

[3] R. Arditi and L.R. Ginzburg, Coupling in predator-prey dynamics: Ratio-dependence, *J. Theor. Biol.* **139** (1989) 311-326.

[4] V.I. Arnol'd, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag 1983.

[5] T.M. Asfaw, Dynamics of Generalized Time Dependent Predator Prey Model with Nonlinear Harvesting, *Int. Journal of Math. Analysis*, **3**(30) (2009), 1473 - 1485.

[6] M.A. Aziz-Alaoui, M.D. Okiye, Boundedness and global stability for a predator-prey model with modified Leslie-Gower Holling-type II schemes, *Appl. Math. Lett.* **16**(7) (2003) 1069-1075.

[7] A.D. Bazykin, F.S. Berezovskaya, G.A. Denisov and YU.A. Kuznetzov, The influence of predator saturation effect and competition among predators on predator-prey system dynamics, *Ecological Modeling*, **14** (1981) 39-57

[8] F. Berezovskaya, G. Karev and R. Arditi, Parametric analysis of the ratio-dependent predator-prey model, *J. Math. Biol.* **43** (2001) 221-246.

[9] H.W. Broer, V. Naudot, R. Roussarie and K. Saleh, A predator-prey model with non-monotonic response function, *Regular & Chaotic Dynamics* **11** (2006), 155-165.

[10] R.P. Gupta and P. Chandra, Bifurcation analysis of modified Leslie-Gower predator-prey model with Michaelis-Menten type rey harvesting, *J. Math. Anal. Appl.* **398** (2013) 278-295.

[11] J. Guckenheimer and P. Holmes, Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector fields, *Appl. Math. Sc.* **42**, Springer-Verlag 1982.

[12] E.J. Doedel, R.A. Champneys, T.F. Fairgrieve, Yu.A. Kuznetsov, B. Sandstede and X.J. Wang, AUTO2000: Continuation and Bifurcation Software for Ordinary Differential Equations (with HomCont), User's Guide, Concordia University, Montreal, Canada 1997-2000. (http//indy.cs.concordia.ca).

[13] W. Govaerts, Y.A. Kuznetsov and A. Dhooge, Numerical Continuation of Bifurcations of Limit Cycles in MATLAB. *SIAM J. Sci Comp.* **27**(2005), 1407-1435.

[14] D. Hanselman, *Mastering Matlab*, University of Maine 2001. (http://www.mathworks.com).

[15] K. Saleh, A Ratio-dependent Predator-prey System with Quadratic Predator Harvesting , *Journal of Asian Transactions on Basic and Applied Sciences*, vol. **02**, no. 04, pp. 21-25, 2012.

[16] K. Saleh, Multiple Attractors Involving Chaos in Predator-prey Model with Non-monotonic Response Function, *Journal of Basic and Applied Sciences*, vol. **11**, no. 4, pp. 56-61, 2011.

[17] Y.A. Kuznetsov, *Elements of Applied Bifurcation Theory*, Springer-Verlag 1995.

[18] D. Xiao, L. Jennings, Bifurcations of a ratio-dependent predator–prey system with constant rate harvesting, *SIAM Appl. Math*. **65** (2005) 737–753.

[19] D. Xiao, W. Li and M. Han, Dynamics in a ratio-dependent predator-prey model with predator harvesting, *J. Math. Anal. Appl.* **324**L (2006) 14-29.

[20] Y. Zhu and K. Wang, Existence and global attractivity of positive periodic solutions for a predator-prey model with modified Leslie-Gower Holling-type II schemes, *J. Math. Anal. Appl.* **384** (2011) 400-408.

[21] F.D. Chen, On a nonlinear non-autonomous predator-prey model with diffusion and distributed delay, *J. Comput. Appl. Math*. 180(1) (2005) 33-49.

[22] P.J. Pal, S. Sarwardi, T. Saha, P.K. Mandal, Mean square stability in a modified Leslie-Gower and Holling-type II predator-prey model, *J. Appl. Math. Inform*, 29(3-4) (2011) 781-802.