New Strain based Triangular Finite Element for the Vibration of Circular Cylindrical Shell with Oblique End

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Abstract— A finite element solution is presented to the problem of natural vibration of circular cylindrical shell with oblique end by using a new strain based triangular cylindrical finite element. The new proposed triangular element is based on assumed strains and has only five necessary degrees of freedom at each corner node. The displacement fields of the element satisfy the exact requirement of rigid body displacement. The efficiency of the developed element is first tested by applying it to the calculation of natural frequencies of simple shell problem. The developed element is further applied to analyze the problem of natural vibrations of a cylindrical shell with oblique end. Two geometric cases are considered. The first case involves a shell with a lower perpendicular end and an upper oblique end and both ends have clamped supports. The second case involves a shell with both ends oblique and clamped. Finally, results from a parametric study for both cases are presented.

Index Term— Strain based shell element, finite element, vibration analysis, cylindrical shell.

1. INTRODUCTION

Thin cylindrical shells with one or two oblique ends are of common use in pressure vessel and piping networks in the form of metric bend nozzles and turnnion pipe support in some engineering constructions, such as aerospace and ship structures. These structures are frequently subjected to dynamic loads and thus an analysis to determine the vibration response of these shells and the effect caused by the geometry aspect in the design of such problems. The most popular and effective method of numerical solutions to such problems was the finite element method.

Considerable attention has been given to applying the finite element method of analysis to curved structures. The early work was devoted to shells of revolution in which closed ring shell segments are used (Jones and Strome, 1966) and also to the development of facet elements (Zienkiewics, 1977) and a curved rectangular element (Connor and Brebbia, 1967). The above elements are based on assumed polynomial displacements and have in common linear representation of the in-plane displacements. Convergence of results was found to be slow and attention was therefore given to the development of high order elements. In Ref. (Bogner et al, 1967) a rectangular cylindrical shell element based on the popular cubic polynomial displacement field was used to represent the out-of-plane and in-plane displacements. It was pointed out (Cantin and Clough, 1968) that elements based on polynomial displacement assumptions do not satisfy the requirement of rigid body modes of displacements and a rectangular cylindrical shell element where the rigid body components of the in-plane and out-of-plane displacements are coupled by trigonometric terms was produced. The work on the development of even higher order elements continued and one of the most successful elements was developed by Cowper et al. (1979), in which the quintic and cubic displacements are assumed for bending and in-plane deformations, respectively. The resulting element has 36 degrees of freedom. In Refs. (Argyris, 1968 and Dawe, 1975) elements using more than 50 degrees of freedom were developed by representing the inplane displacements with higher order polynomial terms to provide an improvement of accuracy. More recently, Groenwold and Standen (1995) developed a four node flat shell quadrilateral element with 6 degrees of freedom per node by using a modified form of the drilling degree of freedom membrane of Ibrahimbegovic et al (1990) and the assumed strain plate element of Bathe and Dvorken (1985). Also Koziy and Mirza (1997) presented a thick shell element with cubic polynomials for displacements and quadratic polynomials for rotations which required the use of reduced integration to provide improved results for thin shells.

It is clearly seen that the developed elements are either flat elements or higher order curved elements associated with additional non-essential degrees of freedom to obtain improved accuracy. It is also well known that flat shell elements require the curved panels to be divided into a large number of elements to obtain satisfactory level of accuracy and are only used due to their simplicity. Similarly the high order elements associated with a large number of degrees of freedom not only lead to a considerable increase in the total number of unknowns to be solved for, but also lead to a much wider bandwidth of the overall structural matrix. Also the additional
internal degrees of freedom are not associated with physical corresponding generalized forces. Therefore, what is needed is a formulation that allows the development of curved high order elements with the minimum required degrees of freedom that are simple to implement but efficient to use. In this respect, the strain based approach was found to lead to high order elements with the minimum nodes and degrees of freedom except for special purposes such as the drilling rotation and was used successfully to develop curved beam and different types of shell elements (Ashwell et al., 1971 and 1972; Sabir et al., 1975, 1979, 1987; El-Erris, 1987, 1994; Mousa, 1998 and Djoudi, 2004). One of the features of the strain based approach is that the method allows the in-plane components of the displacements to be represented by higher order polynomial terms than the out of plane components without increasing the number of degrees of freedom. This is of a particular interest since the improvement obtained by the higher order elements is mainly due to the representation of the in-plane displacement by higher order polynomial terms. Later, Mousa et al (2012) used the strain-based approach to develop a new rectangular cylindrical finite element. This element has only five degrees of freedom at each of four nodes.

To model a cylindrical shell with oblique ends shape by finite elements method, a triangular cylindrical shell element is needed. In the present paper, a new triangular shell element based on assumed strain and satisfies the requirements of rigid body modes is developed and then used to investigate the vibration of cylindrical shell with oblique ends. Finally results from a parametric study are presented.

2. Theoretical Consideration of the Displacement Functions for the New Cylindrical Element

In a system of curvilinear coordinates, the simplified strain-displacement relationship for the cylindrical shell element shown in Figure (1) can be written as:

\[ \varepsilon_x = \frac{\partial u}{\partial x} + \frac{w}{R}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \]

and

\[ K_x = -\frac{\partial^2 w}{\partial x^2}, \quad K_y = -\frac{\partial^2 w}{\partial y^2}, \quad K_{xy} = -2\frac{\partial^2 w}{\partial x \partial y} \]

(1)

Where \( u, v \) and \( w \) are the displacement in the \( x, y \) and \( z \) axes, \( \varepsilon_x, \varepsilon_y \) are the in-plane direct axial and circumferential strains and \( \gamma_{xy} \) is the in-plane shearing strain. Also\( K_x, K_y, K_{xy} \) are the mid-surface changes of curvatures and twisting curvature respectively and \( R \) is the principle radii of curvature.

Equation (1) gives the relationships between the six components of the strain and three displacement \( u, v \), and \( w \). Hence, for such a shell there must exist three compatibility equations which can be obtained eliminating \( u, v \) and \( w \) from equation (1).

This is done by a series of differentiations of equation (1) to yield the following compatibility equations:

\[ \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} + \frac{K_y}{R} = 0 \]

\[ \frac{\partial K_{xy}}{\partial x} - 2\frac{\partial K_y}{\partial y} = 0 \]

\[ \frac{\partial K_{xy}}{\partial y} - 2\frac{\partial K_x}{\partial x} = 0 \]

(2)

To keep the triangular element as simple as possible, and to avoid the difficulties associated with internal and non-geometric degrees of freedom, the essential five degrees of freedom at each corner node are used, namely \( u, v, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \). Thus, the triangular cylindrical element, to be developed has a total of fifteen degrees of freedom and 15x15 stiffness matrix. To obtain the rigid body components of the displacement field, all the strains, as given by equations (1), are set to zero and the resulting partial differential equations are integrated. The resulting equations for \( u, v, \) and \( w \) are given by:

\[ u_1 = -\frac{a_1 x}{R} - \frac{a_2}{2R} \left( \frac{x^2}{2R} - \frac{a_3}{R} + a_4 + a_6 y \right) \]

\[ v_1 = -a_3 \left( \frac{x^2}{2R} \right) + a_5 - \frac{a_6 x}{2} \frac{\partial K_{xy}}{\partial y} - 2 \frac{\partial K_x}{\partial x} \]

(3)

\[ w_1 = a_1 + a_2 x + a_3 y \]

Where \( u_1, v_1, \) and \( w_1 \) are the rigid body components of the displacement fields \( u, v, \) and \( w \), respectively, and are expressed in terms of the six independent constants \( a_1, a_2, ..., a_6 \).

Since the element has fifteen degrees of freedom, the final displacement fields should be in terms of fifteen constants. Having used six for the representation of the rigid body modes, the remaining nine constants are available for expressing the straining deformation of the element. These nine constants can be apportioned among the strains in several ways, for the present element we take:

\[ \varepsilon_x = a_7 - \frac{1}{R} \left( \frac{a_{12}}{2} + \frac{a_{13} x^2}{2} + \frac{a_{14} y^3}{6} \right) \]

\[ \varepsilon_y = a_8 \]

\[ \gamma_{xy} = a_9 \]

\[ K_x = a_{10} + a_{11} x y \]

\[ K_y = a_{12} x + a_{13} y \]

\[ K_{xy} = a_{14} + \left[ a_{11} x^2 + 2 a_{13} y^3 \right] \]

(4)

In which the un-bracketed independent constants terms in the above equations were first assumed. The linking bracketed terms are then added to satisfy the compatibility equation (2).

Equations (4) are then equated to the corresponding six expressions, in terms of \( u, v \) and \( w \) from equations (1) and the resulting equations are integrated to obtain:
The complete displacement functions are the sums of corresponding expressions from eqs. (3) and (5). These can be written as:

\[
\{\delta\} = [G(x, y)] \{a_i\} \tag{6}
\]

The coordinates of each node are then substituted to give:

\[
\{\delta_e\} = [C] \{a_i\} \tag{7}
\]

The constants \{\text{a}_i\} can be obtained as:

\[
\{\text{a}_i\} = [C]^{-1} \{\delta_e\} \tag{8}
\]

Hence, the displacements \{\delta\} within the element can be related to the nodal displacements \{\delta_e\} by the following equation:

\[
\{\delta\} = [G(x, y)] \{a_i\} \tag{9}
\]

where \([G(x, y)] [C]^{-1}\) are the interpolation functions.

Having derived the final expression for the displacement functions, it is now possible to obtain the elemental stiffness and mass matrices which are given by:

\[
K_e = [C^{-1}]^T \left( \iint B^T DB \, dx \, dy \right) [C]^{-1} \tag{10}
\]

and

\[
M_e = \rho \left[ C^{-1} \right]^T \left( \iiint N^T N \, dx \, dy \right) [C]^{-1} \tag{11}
\]

Matrix \([C]\) is the transformation matrix, relating the degrees of freedom at each three corners of the element to the constants \text{a}_i in the shape function, \([B]\) is the strain matrix, \([D]\) is the rigidity matrix which relates the stress resultant to the strains and \([N]\) is the matrix of the polynomial functions of the displacement.

These matrices are then assembled to obtain the total structural stiffness and mass matrices \(K\) and \(M\).

3. CONVERGENCE AND MESH DISCRETIZATION

Convergence test is carried out due to the domain discretization of a simply supported natural frequencies cylindrical panel. For the test, regular grids were adopted with equal number of elements in each direction. The cylindrical panel has the following dimensions and material properties:

\[
L = 1.000m \quad r = 2.000m \quad \alpha = 0.5 \text{ rad} \quad t = 0.005 \text{ m}
\]

\[
E = 208 \times 10^9 \text{N/m}^2 \quad \rho = 7833 \frac{\text{Kg}}{\text{m}^2} \quad \text{and} \quad v = 0.29
\]

The results obtained from the present element are compared with those obtained from Abaqus (1997) and the theoretical solution (Soedel, 1981). The Abaqus results are obtained using the 8 nodes shells element. Figures 2 and 3 shows the convergence of the two lowest frequencies with the degrees of freedom used for the simply supported cylindrical panel. Both the first and the second natural frequencies are associated with one half wave in the axial direction and two and three half-waves respectively in the circumferential direction. It is seen that as the finite element mesh is refined both results tend to converge toward the exact solution with better results obtained from the present element. Abaqus solution required the cylindrical panel to be divided into a large number of elements to obtain satisfactory results.
Convergence study is carried out on the problem. Table 1 and 2 shows the vibration of the results for the lowest natural frequencies with different meshes densities are presented ranging from 6x6 to 16x16 are almost identical and in good agreement with the different quadrature method DQM (Hu and Redekop, 2003). These results show the higher performance of the proposed element. The element is now ready to investigate the effect of some parameters on the natural frequencies of the considered problem.

### Table I

<table>
<thead>
<tr>
<th>Mesh Size</th>
<th>Mode 6x6</th>
<th>8x8</th>
<th>10x10</th>
<th>12x12</th>
<th>14x14</th>
<th>16x16</th>
</tr>
</thead>
</table>

### Table II

<table>
<thead>
<tr>
<th>Mesh Size</th>
<th>Mode 14x14</th>
<th>16x16</th>
<th>18x18</th>
<th>20x20</th>
<th>22x22</th>
<th>24x24</th>
</tr>
</thead>
</table>

A parametric study is conducted for the problem as in the previous section (CC) with same dimension and material properties. As well another case for the same problem but with both ends oblique and clamped (CSYM) is considered. The latter case (CSYM) assumed symmetric about the amid length perpendicular plane. These parameters are, the length ratio, radius ratio and obliquity angle. The length ratio of the cylinder were defined in terms of the ratio of the length to the thickness of the cylinder (L/t) and the range of this ratio considered in between 100 and 400. The radius ratio was defined in terms of the ratio of the radius to the thickness of cylinder (r/t) and ratio considered various from 12.5 to 100. Finally the range of the obliquity angle (α) considered in between 30° to 0°. Figures (4 to 9) show the results for each case considered above.
Frequencies are seen to decrease significantly with increase in the length ratio, and radius ratio. Frequencies also generally decrease with increase in the obliquity angle, but the decrease is smaller.

5. CONCLUSION

A shallow triangular cylindrical shell finite element based on assumed strains has been successfully developed. The element also satisfies the requirement of the free rigid-body modes. The efficiency of the element was demonstrated by applying the calculation of the natural frequencies of cylindrical shell simple problem. This element is then used to analyze the problem of natural vibration of circular cylindrical shell with oblique ends. Parametric studies have been carried out to study the effect of length ratio, radius ratio and obliquity angle on the natural frequencies of the oblique ends problem.

REFERENCES