# Phase Effects in Metamaterials at Third-Harmonic Generation

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***Abstract--*** **In the current work the theory of third-harmonic generation will be developed for the case of metamaterials while the constant-intensity approximation has been used for the phase change of interacting waves. Different parameters are taken into consideration with us for investigation of the intensity of third-harmonics for metamaterials. The effect of self-action, at the presence of phase change, for light waves in metamaterials has been investigated by authors. We compare this effect with an analogous effect in usual homogeneous cubic media. We will prove that the phase velocity of the pump wave could be changed by changing pump intensity, length of nonlinear medium and phase mismatch between interacting waves. The possibility of phase change for fundamental radiation in metamaterials, which has the order of magnitudes higher than homogenous cubic media, will be feasible.**

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***Index Term--*** **Metamaterials, frequency conversion, third-harmonic generation, constant-intensity approximation.**

## 1 Introduction

Metamaterials are currently the important subjects in Physics and electromagnetic communities. They are also subjected to the investigations in the electromagnetics of complex media. These fields are being carried out both from the material characterization and manufacturing viewpoints. One new application for these dielectrics is the area of electromagnetic metamaterials. These materials have the capability of exhibiting effective responses to electric and magnetic fields [1]. The spectrum of the application is from communication to health care, from manufacturing semiconductor to the power generation and conversion and also in military usage [2].

Because of these the properties of these kind of materials are subjected to extensive studies [3-7]. Until this moment authors of [3-7] have been able to use these artificial magnetic materials experimentally in the range that starts from radio frequencies and expands to terahertz [1] and possibly near to the IR frequency. Results of researches in this direction also are given in Refs. 8-12.

One of the important features of the metamaterials is their visibility in certain frequency range so the famous nonlinear effect of generation of harmonics will be a beautiful example of unusual results for the electromagnetic wave interaction in metamaterials [13-17].

According to the paper [5], and also a study in the reference 15, it has been possible to measure a strong optical response in metamaterials with quadratic and cubic nonlinear nature.

We should consider that the condition of phase matching and weak phase mismatch is used for the metamaterials in series of works particularly the constant-field approximation [13-14].This is the kind of approximation in which we take the real amplitude and the phase of initial waves both to be constant. This particular approximation is only valid at the initial stage of nonlinear interaction due to the fact that we can ignore the back action of generated or amplified waves over the intensive pump wave entirely. In this approach though we will lose some important feature of nonlinear process while the constant-intensity approximation, [18-20], will not require any kind of limitation for the phase of interacting waves. This allows us to carry on a more strict analysis of nonlinear interaction of waves for the cases in which the phase change of interacting waves are considered. In this approach the partially reversed influence of excited waves on exciting ones will be considered.

The main requirement, as we know, for proceeding of nonlinear optical process is the necessity of optimum phase relationship between interacting waves. Breaking of these conditions will bring up the mismatch of the phases and as consequence the inefficiency of nonlinear process will decrease. One of the main reasons responsible for infringement of the condition of maximal phase correlation is the phase mismatch.

This paper has cited the results of theoretical investigation for mixing of the fourth waves in metamaterials at the third-harmonic generation while the constant-intensity approximation is considered. Under this approximation it would be possible to observe the peculiar aspect of nonlinear process though it will not be practical to investigate them under the imposed limits of constant-field approximation. We will study the effect of self-action of the light wave in metamaterials to have a comparative scale against the similar effects occurring in the homogenous cubic medium.

## 2 Third-Harmonic Generation in Metamaterials

Following [9, 11, 16], we assume that the metamaterials have the negative value for both the dielectric permittivity and magnetic permeability ($\varepsilon_1 < 0, \mu_1 < 0$) at pump frequency $\omega_1$ and the positive values of dielectric permittivity and magnetic permeability ($\varepsilon_3 > 0, \mu_3 > 0$) at a harmonic frequency $\omega_3 = 3\omega_1$. Assuming that the pump wave that has $S_1$ radiation flow of pump is perpendicular over the right hand side of the surface of metamaterials with length $\ell$ we consider that the light ray will propagate on the negative direction of z-axis.

The direction of the wave vector, $\vec{k}_{1,3}$, is defined by the condition of collinear phase-matching. By considering these conditions the wave vectors in metamaterials, $\vec{k}_{1,3}$, and the vector signifying the flow of energy of the harmonics, $\vec{S}_3$, will be directed along the positive z axis. These three vectors are opposite to Poynting vector $S_1$. Thus, Poynting vector of pump wave $\vec{S}_1$ is counter-directed with respect to harmonics $\vec{S}_3$.

Following the methods mentioned in [9-11,15] the process for the generation of 3[rd] harmonic in metamaterials for the fourth-wave mixing will be given by the reduced equations [16,21]

$$\begin{aligned}\frac{dA_1}{dz}+\delta_1 A_1 &= -i\gamma_1 A_3\left(A_1^*\right)^2\exp(i\Delta z),\\ \frac{dA_3}{dz}+\delta_3 A_3 &= i\gamma_3 A_1^3\exp(-i\Delta z).\end{aligned} \qquad (1)$$

Here $A_{1,3}$ show the complex amplitudes of pump and third-harmonic waves for the frequency of $\omega_{1,3}$, respectively,

$$\gamma_1=\frac{2\pi\,\omega_1}{c}\sqrt{\frac{\mu_1}{\varepsilon_1}}\chi_{eff}^{(3)},\ \gamma_3=\frac{2\pi\,\omega_3}{c}\sqrt{\frac{\mu_3}{\varepsilon_3}}\chi_{eff}^{(3)},$$

The designations $\gamma_{1,3}$ and $\delta_{1,3}$ are the nonlinear coupling coefficients and the wave absorption coefficients of interacting waves in the metamaterial at frequencies $\omega_{1,3}$, respectively, $\Delta = k_3 - 3k_1$ shows the phase mismatch between the interacting waves, $k_{1,3}$ ($k_{1,3}>0$) signify moduli of the wave vectors $\vec{k}_{1,3}$ directed along the $z$ axis and $\chi_{\text{eff}}^{(3)}$ stands for the efficient cubic susceptibility of the metamaterial. The "minus" sign (in contrast to the "plus" sign for traditional medium) at the right side of the first equation of (1) appears because of the differences in the sign of the refractive indices $n_{1,3}$ for metamaterials.

When the pump wave $S_1$ is propagating in the metamaterial opposite to the $z$ axis, it follows from the expression for $S_1$ that wave vector $k_1$ is directed towards the side opposite to vector of the energy flow $S_1$, i.e. along the $z$ axis. When the phase matching conditions are considered, then the harmonic wave, generated in nonlinear media, will have a wave vector $k_3$ which coincides with $k_1$ in direction.

Since $\varepsilon_3>0,\ \mu_3>0$ the vectors $k_3$ and $S_3$ will have same direction and will propagate along the z axis so we consider that the system mentioned in (1) has the following boundary conditions for the case where the negative values of dielectric permittivity and magnetic permeability are considered at the pump frequency $\omega_1$ and the positive-values of dielectric permittivity and magnetic permeability are considered at the harmonic frequency $\omega_3 = 3\omega_1$

$$A_1(z=\ell)=A_{1\ell}\exp(i\varphi_{1\ell}),\qquad A_3(z=0)=0. \qquad (2)$$

Here, $z=\ell$ corresponds to an input in a metamaterial from the right side (in the negative $z$ axis direction), $\varphi_{1\ell}$ is the initial phase of pump wave at the input of the nonlinear medium.

After solving the system (1) in the constant-intensity approximation, $I_1(z)=I_1(z=\ell)=I_{1\ell}$, while the boundary conditions are also considered, we will get the complex amplitude for the harmonic wave at the length of z for nonlinear medium ($\delta_{1,3}=0$)

$$A_3(z)=-\frac{\gamma_3 A_{1\ell}^3\cdot\sinh\lambda z}{\frac{\Delta}{2}\sinh\lambda\ell+i\lambda\cosh\lambda\ell}\exp\left[i3\varphi_{1\ell}-i\Delta(z+\ell)/2\right],\ \text{at}$$

$$3\Gamma^2\geq\frac{\Delta^2}{4} \qquad (3)$$

Here

$$\lambda^2=3\Gamma^2-\frac{\Delta^2}{4},\quad \Gamma^2=\gamma_1\gamma_3 I_{1\ell}^2,\quad I_j=A_jA_j^*.$$

And for the situation in which $3\Gamma^2<\frac{\Delta^2}{4}$, we will get

$$A_3(z)=\frac{i\gamma_3 A_{1\ell}^3\cdot\sin\lambda' z}{\lambda'\cos\lambda'\ell-\frac{i\Delta}{2}\sin\lambda'\ell}\exp\left[i3\varphi_{1\ell}-i\Delta(z+\ell)/2\right], \qquad (4)$$

where

$$\lambda'^2=\frac{\Delta^2}{4}-3\Gamma^2.$$

We will have different analytical expression for the conversion of frequency of the metamaterials than the case of ordinary metamaterials [22]. The difference is due to the additional terms in denominator which is proportional to the parameter $\cos\lambda'\ell$ and $\frac{i\Delta}{2}\sin\lambda'\ell$ [connected with the boundary conditions in metamaterials (2)] so the dynamic process of frequency conversion in materials that have negative index will be directly dependent on the complete length of the material, $\ell$, as a result (this fact has been discussed in [11,14]).

We can see from this expression that the amplitude of the harmonic wave is related on a factor which is the result of the back action of the excited harmonic wave on the pump wave. This factor affects the phase of third-harmonic wave. Besides that we will have from (3) [or (4)] that the phase of the harmonic wave, $A_3(z)$, depends on the intensity of the pump. Such result will not be observed in the case of constant-field approximation.

At $\gamma_1 = 0$ from (3) [or (4)] we can see the lack of pump depletion which means the case of constant-field approximation.

From the system (1) with use the expression (3) for complex amplitude of the fundamental radiation, we arrive at ( $\Gamma^2 > \frac{\Delta^2}{12}$ )

$$A_1(z) = A_{1\ell} \sqrt[3]{\frac{\lambda \cosh \lambda z - i\Delta \cdot \sinh \lambda z / 2}{\lambda \cosh \lambda \ell - i\Delta \cdot \sinh \lambda \ell / 2}} \exp\left[ i\varphi_{1\ell} - i \frac{\Delta(\ell - z)}{6} \right] \quad (5)$$

For the fundamental-radiation intensity, from (5) we obtain

$$I_1(z) = I_{1\ell} \sqrt[3]{\left( \lambda^2 \cosh^2 \lambda z + \frac{\Delta^2}{4} \sinh^2 \lambda z \right) / \left( 3\Gamma^2 \cosh^2 \lambda \ell - \frac{\Delta^2}{4} \right)} \quad (6)$$

We will get the following, considering the relation (3), for the conversion of the energy of pump wave into the energy of the harmonic wave (or the given intensity of the third-harmonic wave)

$$\eta_3(z) = I_3^{'} = \frac{I_3(z)}{I_{1\ell}} = \gamma_3^2 I_{1\ell}^2 \frac{\sinh^2 \lambda z}{3\Gamma^2 \cosh^2 \lambda \ell - \frac{\Delta^2}{4}} \quad \text{at } \Gamma^2 > \frac{\Delta^2}{12}. \quad (7)$$

In the case of $\Gamma^2 < \frac{\Delta^2}{12}$, we have

$$\eta_3(z) = I_3^{'} = \frac{I_3(z)}{I_{1\ell}} = \gamma_3^2 I_{1\ell}^2 \frac{\sin^2 \lambda z}{\frac{\Delta^2}{4} - 3\Gamma^2 \cos^2 \lambda \ell}. \quad (8)$$

We can see from (7) and (8) that the optimum values exist for the efficiency of frequency conversion for the fundamental radiation intensity, nonlinear medium length and phase mismatch in which the efficiency of conversion is maximal.

A typical spatial modulation is observed for the case of harmonic waves traveling in a medium with positive refractive index. The coherent length of nonlinear interaction, for the case of constant-field approximation, is derived from the condition of $\Delta \ell_{coh}^{CFA} = \pi$ . Considering the phase effects in the constant-intensity approximation, in the case of a homogeneous cubic medium we derive the expression $\ell_{coh\ \text{homogeneous}}^{CIA} = 0.5\pi / (3\Gamma^2 + \frac{\Delta^2}{4})^{1/2}$. In the metamaterials this expression has the form of $\ell_{coh}^{CIA} = 0.5\pi / (3\Gamma^2 - \frac{\Delta^2}{4})^{1/2}$ at $\Gamma^2 > \frac{\Delta^2}{12}$ and in the case $\Gamma^2 < \frac{\Delta^2}{12}$, we have $\ell_{coh}^{CIA} = 0.5\pi / (\frac{\Delta^2}{4} - 3\Gamma^2)^{1/2}$. As the intensity of incoming radiation at the fundamental frequency increase the constant-intensity approximation allows us to obtain a more correct expression for the length of coherence for the items of the same order $(3\Gamma^2 \sim \frac{\Delta^2}{4})$ .

Considering (7) and (8) we can find the optimum value of phase mismatch, $\Delta^{opt}$, via the numerical solution of the following equation(particularly in the condition of $\Gamma^2 < \frac{\Delta^2}{12}$ )

$$3(\Gamma z)^2 \cosh^2 \lambda \ell - \left( \frac{\Delta z}{2} \right)^2 = (\frac{3\Gamma z}{2} \Gamma \ell \cdot \sinh 2\lambda \ell + \lambda z) \tanh \lambda z \quad (9)$$

and $I_{1\ell}^{opt}$ is achieved by numerical solving of the following equation and $I_{1\ell}^{opt}$ is obtained by also numerical solving of the equation

$$\left[ 9\Gamma \ell \sinh 2\lambda \ell + 2 \frac{\lambda}{\Gamma} \left( \frac{\Delta}{2\Gamma} \right)^2 \right] \tanh \lambda z = 6\Gamma z \left[ 3\cosh^2 \lambda \ell - \left( \frac{\Delta}{2\Gamma} \right)^2 \right] \quad (10)$$

From the equations (9) and (10) it follows that the values $\Delta^{opt}$ and $I_{1\ell}^{opt}$ depend not only on the interaction length but also the coefficients of nonlinear interaction in the metamaterials. The value of $\Delta^{opt}$ depends on the pump intensity, while $I_{1\ell}^{opt}$ depends on the phase mismatch.

The efficiency of frequency conversion $\eta_3$ under the condition of phase matching ( $\Delta = 0$ ) reads

$$\eta_3(z) = \frac{I_3(z)}{I_{1\ell}} = \gamma_3^2 I_{1\ell}^2 \frac{\sinh^2 \lambda'' z}{3\Gamma^2 \cosh^2 \lambda'' \ell}. \quad (11)$$

Here $\lambda''^2 = 3\Gamma^2$.

We can calculate the efficiency of conversion of frequency, $\eta_3$ at $\Delta^2 / 4 = 3\Gamma^2$, by expanding the function in Taylor series around zero ( $\lambda = 0$ ). As a result we will get the following analytical relation

$$\eta_3(z) = \frac{(\Gamma z)^2}{1 + 3(\Gamma \ell)^2}. \quad (12)$$

This implies that at low values of reduced pump intensity, $I_{1\ell}' = \Gamma \ell$ ( $\Gamma z \ll 1$ ), the efficiency of frequency conversion will be proportional to $I_{1\ell}^2$. At greater values of the pump intensity ( $\Gamma z \geq 1$ ) the efficiency of conversion of frequency

$\eta_3$, will not depend on $I'_{1\ell} = \Gamma\ell$. We can also observe the similar dependency on phase mismatch.

**Results and Discussion.**

In figure 1-9 the results of numerical calculations of analytical expressions (5), (6), (10) and (11), under the constant-intensity approximation are presented. The curves demonstrate the dynamics of conversion to the third-harmonic in non-dissipating medium of the metamaterial.

Fig. 1 shows the dependencies of the reduced intensities of the pump wave $I_1^{'} = I_1 / I_{1\ell}$ and third-harmonic $I_3^{'} = I_3 / I_{1\ell}$ on reduced length of the metamaterial $z' = \Gamma z$ for different small values of reduced phase mismatch $\Delta' = \Delta / 2\Gamma$ ($\Gamma^2 > \frac{\Delta^2}{12}$). The differences from the norm of usual dependency of the effectiveness of conversion efficiency, $\eta_3(z)$, is totally obvious for natural materials. The curves strongly differ from the usual dependence of the efficacy of frequency conversion $\eta_3(z)$ observed in natural materials. A monotonous behavior of dependence is observed for the case of materials with negative refraction at small values of the phase matching ($\Gamma^2 > \frac{\Delta^2}{12}$) while for the cubic nonlinear media, at all values of the phase matching, the maximum will be observed. The metamaterials will behave like mirror to reflect the harmonic waves when the condition $\Delta < 2\sqrt{3}\Gamma$ is fulfilled and this will cause for them to have the maximum efficiency of frequency conversion at the input of the metamaterial $(z = \ell)$. In other words, excited radiation of harmonic is directed against the exciting pump wave.

From the curves at the same value of $\Delta'$ we can see that the conversion efficiency for phase matching will be the greatest (this will be evident by comparing solid, dotted and dashed curves in 1 and 2).

Fig. 2 at the same time shows the dependences of $I_1^{'}$ and $I_3^{'}$ on the reduced length of metamaterial $z' = \Gamma z$ for three values of reduced full length of metamaterial $I_{1\ell}^{'} = \Gamma\ell$. By decreasing the full length, $\Gamma\ell$, the conversion efficiency will also decrease tremendously as expected with reduction of full length conversion efficiency.

Fig 3 shows the dependencies of the reduced intensities of the pump wave and the third-harmonic wave on the reduced length of the metamaterial $z'$ for parameter $\Delta'$ =0.5 calculated under the constant-intensity approximation (curves 1 and 2), where a comparison with the data obtained in the constant-field approximation (curves 3 and 4) is also presented. We can see that by taking the back action of the excited harmonic wave on the fundamental-radiation wave into account, the intensities of the pump and harmonic wave will decrease (compare curve 1 with curve 2 and curve 3 with curve 4).

Let's compare our results with the results obtained from [16]. For the case in which the phase matching is established and the length is $\Gamma\ell$ = 1 for material with negative index, we carry out calculations for the efficiency of conversion depending on reflected waves while taking constant-intensity approximation into consideration [see (5)]. The efficiency $\eta_3$ reaches the value 0.3 (solid curve 1 in Fig. 3). For $\gamma_1$ = 0, from (5) we get $\eta_3$ = 0.999 (which is shown by solid curve 5 in Fig. 3). There is good agreement between our results with the results of [16] where the value of 0.39 was obtained for the efficiency of frequency conversion beyond the constant-field approximation.

Fig. 4 shows the dependency of $\eta_3$ over the reduced length of the metamaterial, $z' = \Gamma z$, under the constant-intensity approximation for different value of reduced phase mismatch $\Delta' = \Delta / 2\Gamma$. The dependencies of $\eta_3$ as function of reduced length of the metamaterial $z' = \Gamma z$ calculated in the constant-intensity approximation are presented for different values of reduced phase mismatch $\Delta' = \Delta / 2\Gamma$ in Fig. 4. According to [17], conversion efficacy has oscillating character for large values of the phase matching or at low pump intensities for which the condition $\Delta > 2\sqrt{3}\Gamma$ is implemented. We can see from figure that the period of oscillations of $\eta_3$ (and coherence length) depends on the phase mismatch, pump intensity (via the expressions of $\Gamma$) and complete length of the metamaterial. As a result, in Fig. 5, the dependencies of the reduced coherent length of the metamaterial over the reduced phase matching $\Delta' = \Delta / 2\Gamma$ and under the constant-field and constant-intensity approximations have been shown. We can see from the expression ($\Delta\ell_{coh}^{CFA} = \pi$) that monotonous decrease of the coherent length will occur in the case of the constant-field approximations. In contrast to the result obtained in the constant-field approximation a significant maximum is observed in the vicinity of zero of parameter $\lambda$ (at $\Delta > 2\sqrt{3}\Gamma$) in the case of the constant-intensity approximation. This maximum can be explained by the "minus" sign (in contrast to the "plus" sign for traditional medium) under the root in the expression for $\ell_{coh}^{CIA}$. The "minus" appears because of the different signs of the refractive indices $n_{1,3}$ for metamaterials.

In Fig. 6a, the dependencies of the efficiency of frequency conversion $\eta_3$ on reduced phase mismatch for fourth values of parameter $\Gamma z$ have been shown. By comparing curves in this figure we see that at the phase mismatch process, for the higher values of the reduced length ($\Gamma z$) of the nonlinear metamaterial, the efficacy of frequency conversion will be larger. Besides that the curves demonstrate that by increasing the nonlinear interaction length, increasing of conversion efficacy will take place (compare curves 1-4). At the same time, it is seen from Fig. 6b that as the pump intensity gets larger the conversion efficiency gets higher. With increase in the pump intensity, there takes place increasing of conversion efficacy (compare curves 1-3). From (11) at $\Gamma z = 0.95$ and $\Gamma\ell = 1$, $\eta_3$ =0.24431 is obtained for conversion efficiency which agrees with the numerical solving of the equation (5) (see Fig.6a, curve 3).

Fig.7.a shows the dependency of $\eta_3$ over the reduced pump intensity $I'_{1\ell} = \Gamma z$ for fourth small values of the reduced phase mismatch $\Delta'$ ($\Delta < 2\sqrt{3}\Gamma$) under the constant-intensity approximation. We can see that the metamaterials have greater conversion efficiency at their input $(z = \ell)$ corresponding to the case of phase matching and by increasing the phase mismatch, $\Delta'$, the efficacy of frequency conversion will fall at the output of the nonlinear medium ( $(z = \ell)$). With increase in the phase-mismatch the lateral maximum of the curve for $\eta_3$ are shifted to the direction of greater values of fundamental-radiation intensity (see curves 2-4). On the other hand, it is seen from Fig. 7b that, a pronounced maximum is observed at large values of the phase matching ($\Delta > 2\sqrt{3}\Gamma$) (curves 1-4). The behavior of these curves is similar to the behavior of corresponding curves of Fig. 6a. Thus, conversion efficacy is of oscillating character at large values of the phase matching or at low pump intensities, for which the condition $\Delta > 2\sqrt{3}\Gamma$ is implemented. This fact was noted in [17].

Qualitatively new effects will arise by taking into consideration the back reaction of the harmonic wave on the phase of the fundamental radiation (which is not ubiquitous in constant-field approximation). First of all the location of the minima of harmonic intensity will depend on the pump intensity. Besides that by increasing the phase mismatch the lateral maximum of the curve will be shifted to the direction that has greater values of fundamental-radiation intensity. And finally, as the fundamental radiation intensity increases the maximum located at the center of the curves will be reduced.

## 3 Self-Action Effect

The self-action effect is usually connected with the cubic nonlinearity of media. Let's say that $A_j(z) = a_j(z)\exp\left[i\varphi_j(z)\right]$, by $j = 1, 3$ the system of equations (1) can be rewritten as follows

$$\frac{da_1}{dz} + ia_1\frac{d\varphi_1}{dz} = -i\gamma_1 a_1^2 a_3 \exp\left[i(\varphi_3 - 3\varphi_1 + \Delta z)\right]. \qquad (13)$$

In the constant-intensity approximation, we obtain

$$\frac{d\varphi_1}{dz} = -\gamma_1 a_3 a_1 \cos\left(\varphi_3 - 3\varphi_1 + \Delta z\right) \qquad (14)$$

Considering (4) we can derive the relationship for the real amplitude $a_3(z)$ and $\varphi_3(z)$ for the third-harmonic in the metamaterial and put them to the relation (13). We will be simply able to obtain the change of the pump-wave phase in the process of propagation of light through metamaterials.

$$\varphi_1(z) = \varphi_{1\ell} + \frac{1}{\cosh^2 \lambda l} \cdot \frac{\frac{\Delta}{\lambda}\sinh 2\lambda z - \Delta z}{\left(\frac{\Delta}{\Gamma}\right)^2\left(\tanh^2 \lambda l - 1\right) + 12}. \qquad (15)$$

This should also be mentioned that the expression that shows the change in the phase of the pump wave during the propagation of light in homogenous cubic medium will include the following parameters: $\Delta^{vol}$ is going to be the phase mismatch during the generation of third-harmonics for the regular cubic media, $\lambda^{vol} = \sqrt{3(\Gamma^{vol})^2 + \frac{(\Delta^{vol})^2}{4}}$, $(\Gamma^{vol})^2 = 3\gamma_1^{vol}\gamma_3^{vol} I_{1\ell}^2$ and $\gamma_1^{vol}$ and $\gamma_3^{vol}$ are the nonlinear coupling coefficients of the pump wave and the third-harmonic in the homogeneous cubic medium. Here and below, we will note the process considered in the volume of the nonlinear homogeneous medium by "vol".

Expression for $\varphi_1(z) - \varphi_{1\ell}$ can also be easily obtained from (15). To do this one can simply take the change in boundary condition (2) into consideration $A_3(z = \ell) = 0$ and the change in the sign of the parameter $\Gamma$ that signifies the switching of the metamaterials into the regular homogenous cubic medium $\gamma_1^{vol} = \gamma_1$ and $\gamma_3^{vol} = -\gamma_3$.

Following the expression (15) will tell us that the phase velocity in metamaterials for excited waves and the refraction indices of the medium will depend on $I_{1\ell}$ which is the intensity and this means the self-action of the light wave is observed.

From (15) at $\gamma_1 = 0$ ($\Gamma^2 = 3\gamma_1\gamma_3 I_{1\ell}^2$), the known result of the constant-field approximation will be obtained which is $\varphi_1(z) = \varphi_{1\ell}$ and this means that the pump-wave phase will be constant.

**Results and Discussion.** Fig. 8 and 9 show the dependencies of the phase shift of the pump waves phase $\varphi_1(z) - \varphi_{1\ell}$ in the metamaterials as a function of the intensity of the fundamental radiation and the length of nonlinear medium, respectively, for different values of the phase mismatch. As $\Delta$ increase the absolute value of the difference $\varphi_1(z) - \varphi_{1\ell}$ will increase (this could be seen by comparing curve 1 and 2 in Figs. 8 and 9).

For the cubic and homogenous medium we have the dotted curve plotted on Figs. 8 and 9. By comparing the dependency for both types of media we can see that the phase of the pump will change more for the case of metamaterial (you can see this by comparing the solid and dotted curve from both diagrams). For example for the case of metamaterials, $\varphi_1(z) - \varphi_{1\ell}$ will increase almost five times when $\frac{\Delta}{2\Gamma}$ varies from 0.1 (solid curve 1) to 0.5 (solid curve 2) at

$\Gamma z = 1$ (Fig. 8) while for the homogenous cubic medium this will be almost 26.4 times.

The negative refracting index for metamaterials shows the change of difference in the sign of $\varphi_1(z) - \varphi_{1\ell}$ while converting to general cubic medium.

Numerical calculation shows the same behaviour, in a wide range of parameters, for the curves, $\varphi_1(z) - \varphi_{1\ell}$, for both intensity and length.

This study shows, considering the efficiency of frequency conversion, that materials which have negative refractive index acts like mirror for the incoming radiation for the case of small values of the phase matching or for higher pump intensities under the implementation of the condition $\Delta < 2\sqrt{3}\Gamma$ while the mirror will reflect the harmonic radiation and not the fundamental ones. Due to nonlinear interaction occurring in the media, the wave of the fundamental frequency enters first and then it converts to the third-harmonic wave. Then, as result, the harmonic wave propagates in the opposite direction of incoming radiation and this is the same as the reflection of incoming radiation at the entry of metamaterials at tripled frequency.

## Conclusion

In this work the harmonic generation in a medium with cubic nonlinearity has been investigated. We carried on this study following the condition in which the frequency of the pump wave is negatively refracting and at a tripled frequency of the pump wave it is positively refracting. The case of phase matching and mismatching both has been considered. The theoretical expression, obtained in the constant - intensity approximation, has been solved analytically and we have shown that by considering the phase change of interacting waves a decrease in the efficiency of conversion of frequency will be observed which is accompanied by the change in the phase matching curve parameters versus the pump intensity.

This observation seems to be handy for the sake of correcting the calculations that was done for the metamaterials. We have shown that the propagation of light in metamaterials, which has cubic nonlinearity, has also the self-action effect. The velocity of the pump wave can be depended on the pump intensity, the length of the nonlinear medium and the phase mismatch between the interacting waves. We have demonstrated that the change of the phase of fundamental radiation will be higher than the case of usual homogenous cubic media.

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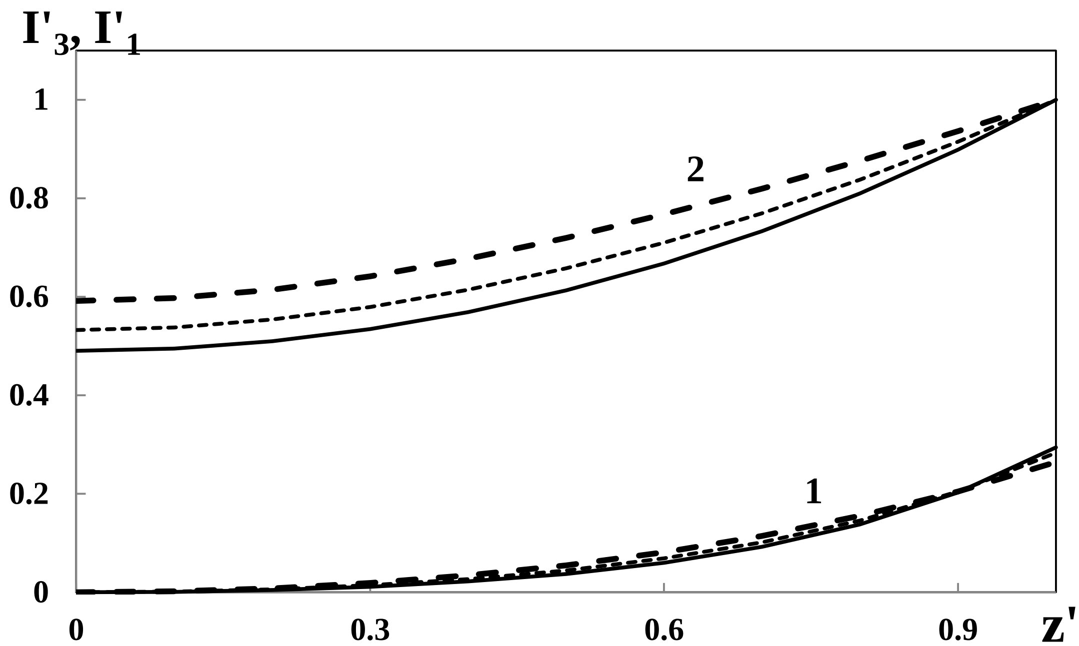


Fig. 1. Dependencies of the reduced intensities of the pump wave $I_1^{'} = I_1 / I_{1\ell}$ (curves 2) and third-harmonic wave $I_3^{'} = I_3 / I_{1\ell}$ (curves 1) on the reduced length of the metamaterial $z' = \Gamma z$ calculated in the constant-intensity approximation for $\Gamma\ell = 1$ at different values of the reduced phase mismatch $\Delta' = \Delta / 2\Gamma$ equal to 0 (solid curves), 1 (dotted curves) and 1.5 (dashed curves).

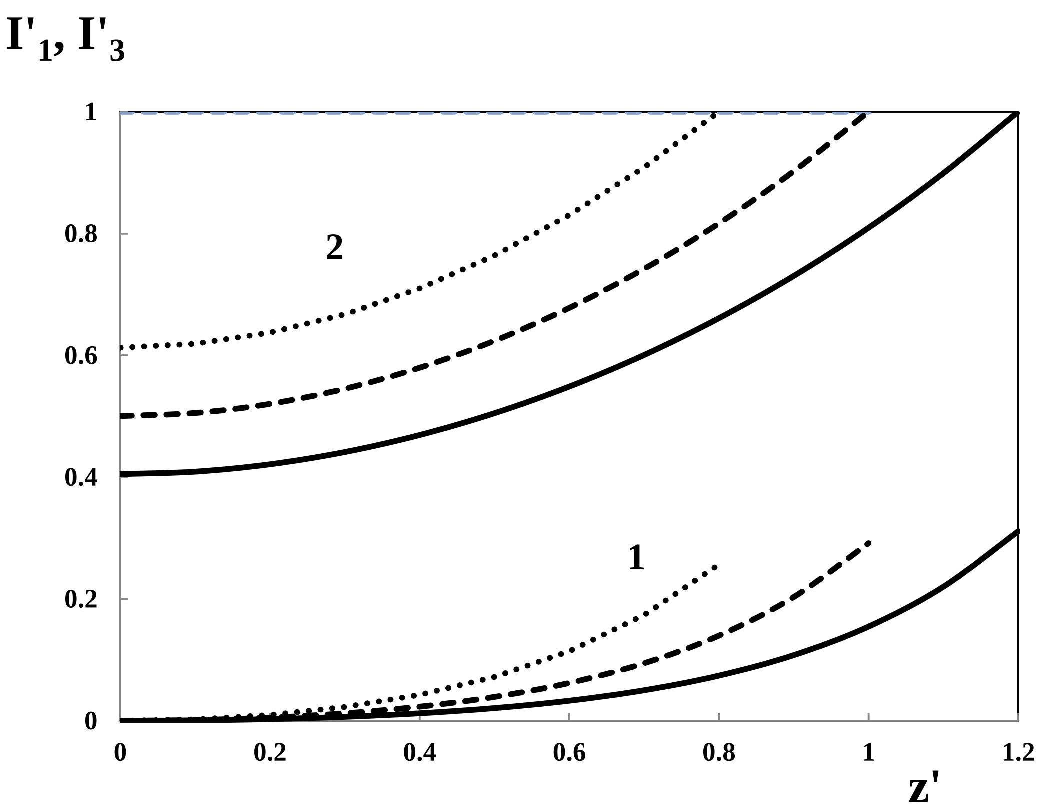


Fig. 2. Dependencies of the reduced intensities of the pump wave $I_1^{'} = I_1 / I_{1\ell}$ (curves 2) and third-harmonic wave $I_3^{'} = I_3 / I_{1\ell}$ (curves 1) on the reduced length of the metamaterial $z' = \Gamma z$ calculated in the constant-intensity approximation at $\Delta' = \Delta / 2\Gamma = 0.5$ for three values of reduced full length of metamaterial $I_{1\ell}^{'} = \Gamma\ell$ equal to 0.8 (dotted curves), 1 (dashed curves) and 1.2 (solid curves).

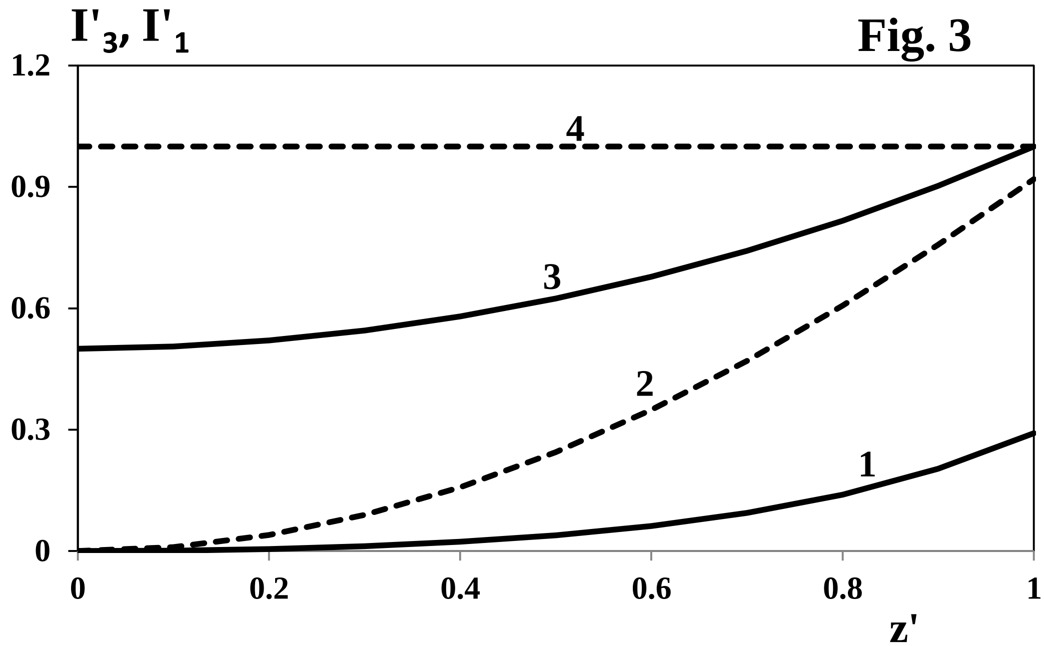


Fig. 3. Dependencies of the reduced intensities of the pump wave $I_1^{'} = I_1 / I_{1\ell}$ (curves 3 and 4) and the third-harmonic wave $I_3^{'} = I_3 / I_{1\ell}$ (curves 1 and 2) on the reduced length of the metamaterial $z'$ calculated in the constant-intensity approximation (solid curves 1 and 3) and the constant-field approximation (dashed curves 2, 4 and solid curve 5) for parameters $\Gamma\ell$ =1 and $\Delta' = \Delta / 2\Gamma = 0$ (curves 1 and 5) and 0.5 (curves 1-4).

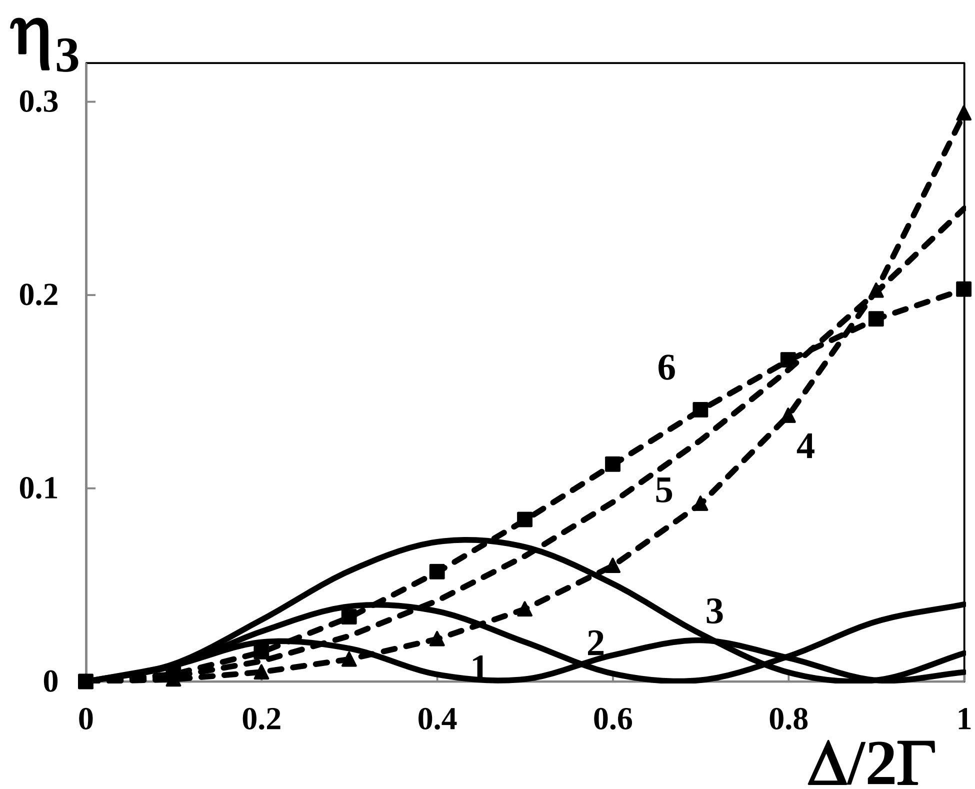


Fig. 4. Dependencies of the efficiency of frequency conversion $\eta_3$ on reduced length of the metamaterial $z' = \Gamma z$ calculated in the constant-intensity approximation at $\Gamma\ell$ =1 for different values of the reduced phase mismatch $\Delta' = \Delta / 2\Gamma$ equal to 7 (curve 1), 5 (curve 2), 4 (curve 3), 0 (curve 4), 1.8 (curve 5) and 2.2 (curve 6).

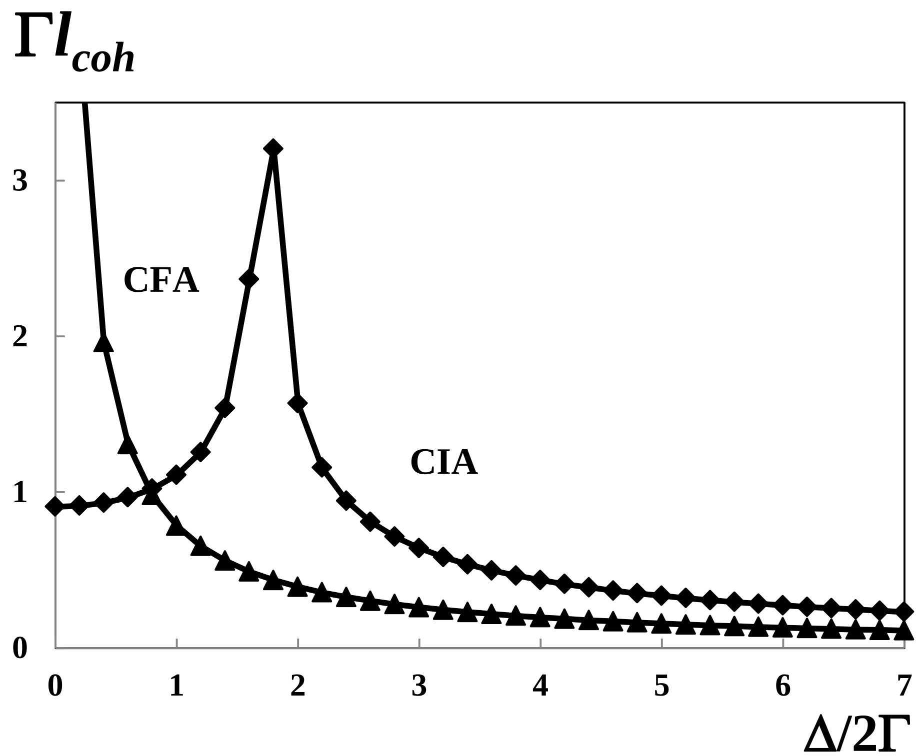


Fig. 5. Dependencies of the reduced coherent length of the metamaterial on reduced phase matching $\Delta' = \Delta / 2\Gamma$ calculated in the constant-field and constant-intensity approximations at $\Gamma\ell$ =1.

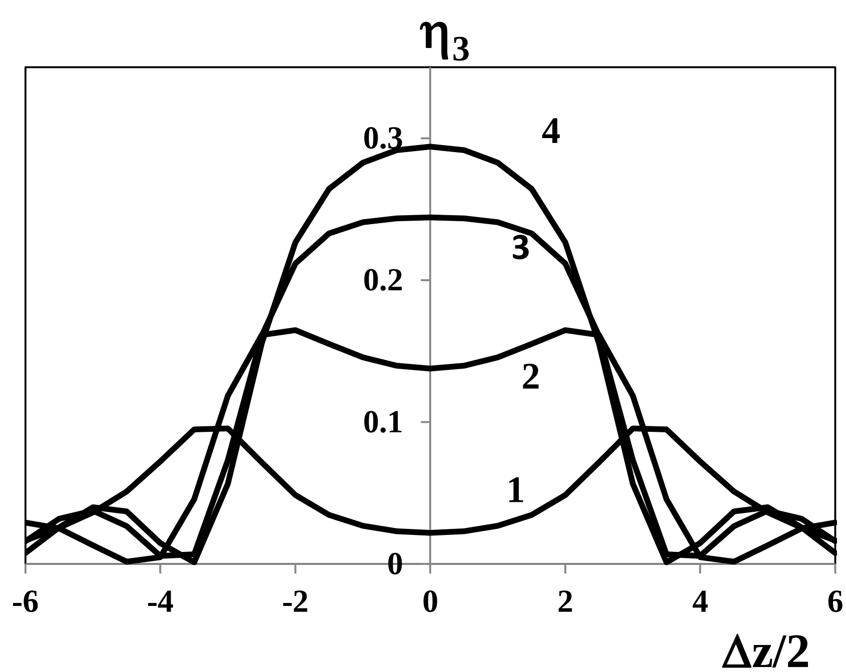


Fig. 6a. Dependencies of the efficiency of frequency conversion, $\eta_3$ on reduced phase mismatch between the wave at the fundamental and third-harmonic frequencies calculated in the constant-intensity approximation at $\Gamma\ell$ =1 for fourth values of parameter $\Gamma z$ equal to 0.4 (curve 1), 0.8 (curve 2), 0.95 (curve 3) and 1 (curve 4).

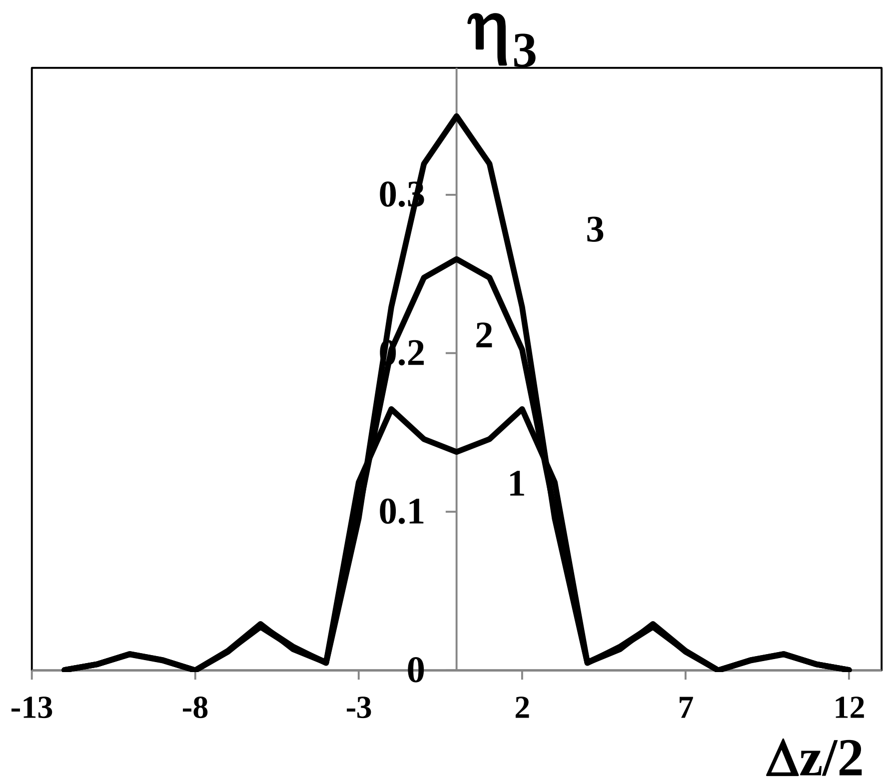


Fig. 6b. Dependencies of the efficiency of frequency conversion, $\eta_3$ , on reduced phase mismatch between the wave at the fundamental and third-harmonic frequencies calculated in the constant-intensity approximation at $\Gamma z$ =0.8 for the following three values of parameter $\Gamma\ell$ equal to 0.7 (curve 3), 0.8 (curve 2) and 1 (curve 3).

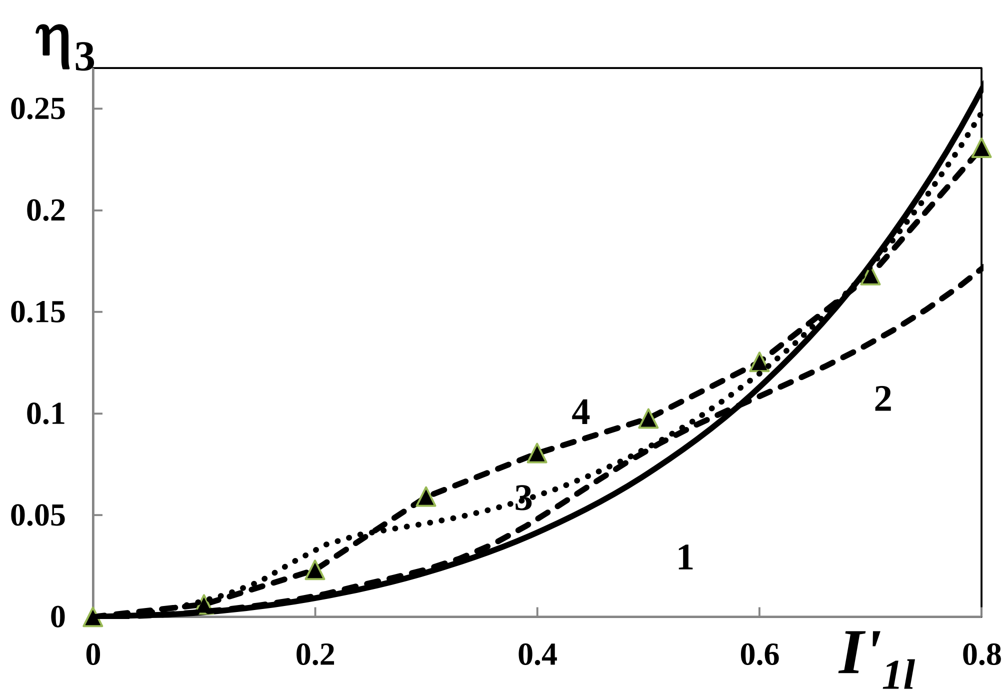


Fig. 7a. Dependencies of $\eta_3$ as function of reduced pump intensity, $I'_{1\ell} = \Gamma z$, calculated in the constant-intensity approximation at $\Gamma\ell$ =0.8 for small values of reduced phase mismatch $\Delta'$ ( $\Delta < 2\sqrt{3}\Gamma$ ) equal to 0 (solid curve 1), 1.9 (dashed curve 2), 0.8 (dotted curve 3) and 1.2 (marked dashed curve 4).

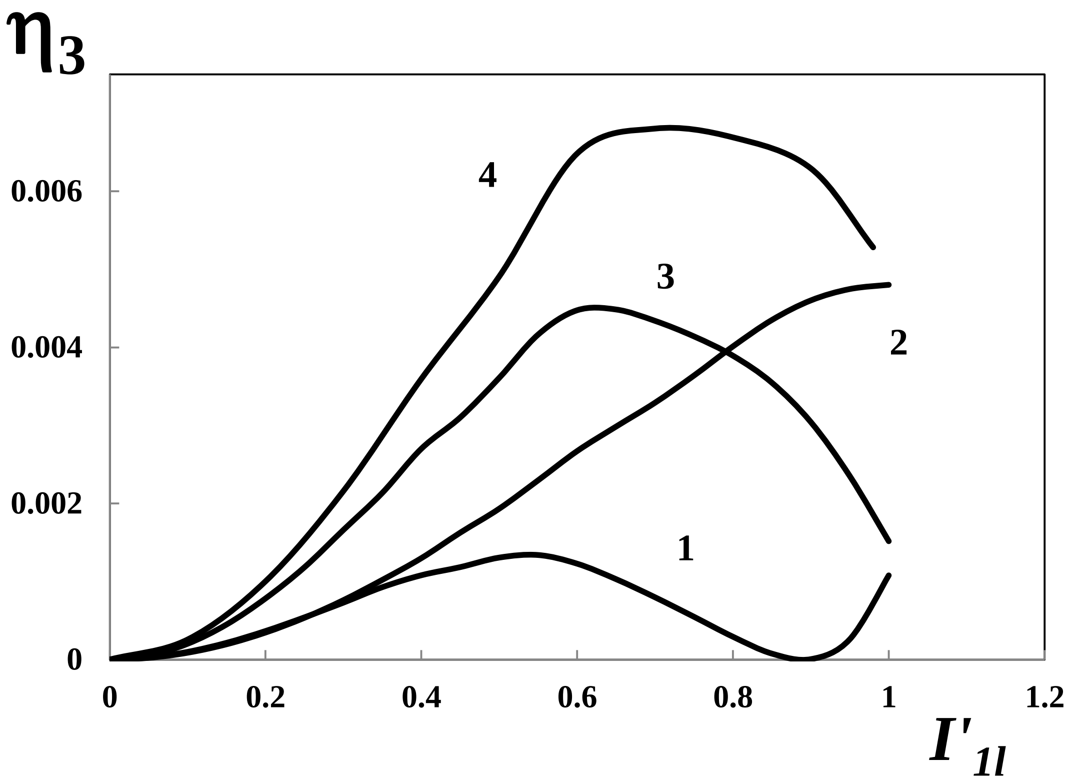


Fig. 7b. Dependencies of $\eta_3$ as function of reduced pump intensity, $I'_{1\ell} = \Gamma z$, calculated in the constant-intensity approximation at $\Gamma\ell = 1$ for large values of reduced phase mismatch $\Delta'$ equal to 3.5 (curve 1), 7 (curve 2), 4 (curve 3) and 3.8 (curve 4).

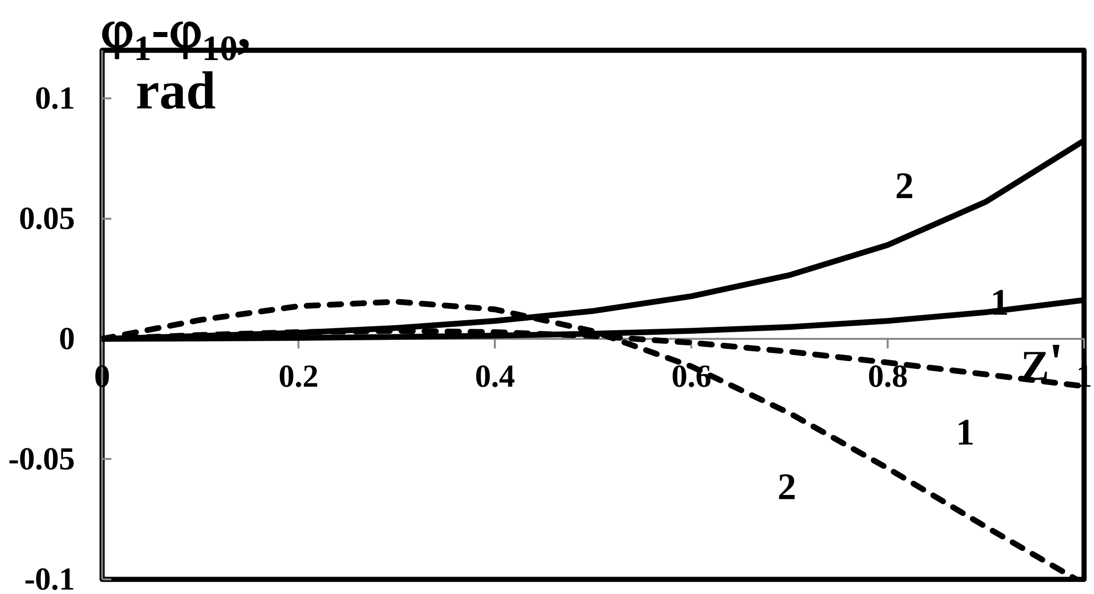


Fig. 8. Dependencies of the phase shift of the pump wave phase, $\varphi_1(z) - \varphi_{1\ell}$ ,in the metamaterial (solid curves 1 and 2) and in the homogeneous medium (dashed curves 1 and 2) as function of the reduced length at $\Gamma\ell = 1$ for the following values of the phase mismatch $\Delta' = \Delta / 2\Gamma = 0.1$ (solid and dashed curves 1) and 0.5 (solid and dashed curves 2).

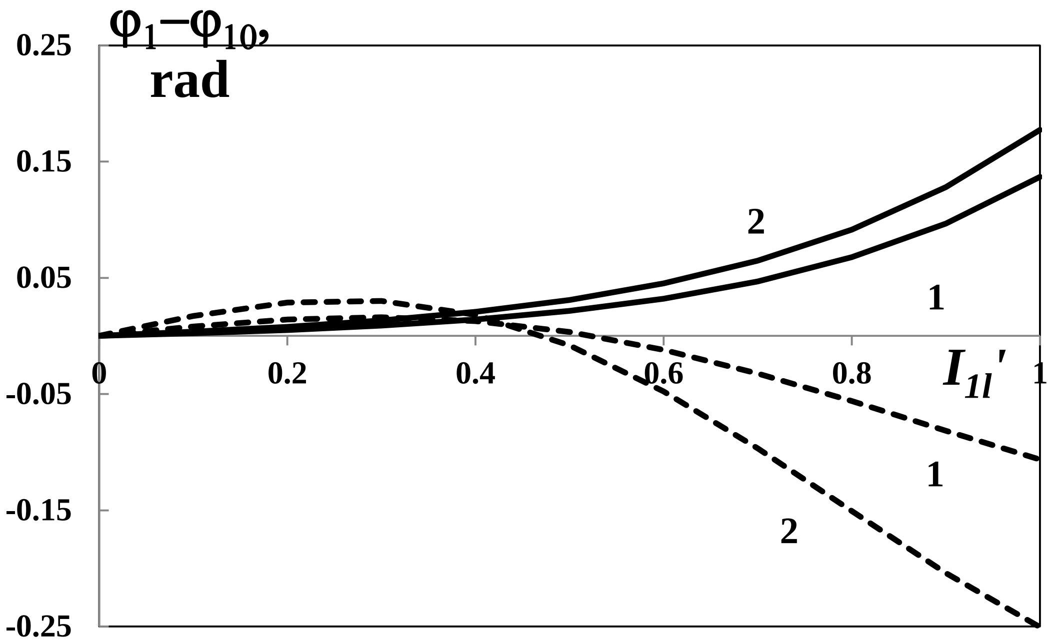


Fig. 9. Dependencies of the phase shift of the pump wave phase, $\varphi_1(z)-\varphi_{1\ell}$, in the metamaterial (solid curves 1 and 2) and in the homogeneous medium (dashed curves 1 and 2) as function of intensity of the fundamental radiation at $\Gamma\ell$ =1 for different values of the phase mismatch $\Delta' = \Delta / 2\Gamma = 0.5$ (solid and dashed curves 1) and 1 (solid and dashed curves 2).