Resonant Buildup of Gravitational Waves

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***Abstract* – Considering so far oscillations of flows bounded by rigid walls .Oscillations of a type – gravitational waves- are produced in flows with free boundaries in the presence of a gravity force. Owing to the presence of the free boundary. Showing below that gravitati-onal waves may turn out to be unstable even if the flow velocity profile of an ideal liquid has no inflection points.**

***Index Term*-- Resonant buildup, Gravitational waves,Surface waves.**

I. INTRODUCTION

The theory of oscillations and stability of continuous media (liquid, gravitating medium) has been the subject of many studies. Most touch upon, and in many cases consider in detail, a continuous medium moving with variable velocity in space [1-12]. The present paper differs in that it employs consistently a point of view according to which the change (growth or damping) of the perturbation amplitudes in a nonuniformly moving medium is due to resonant interaction with the motion of the medium.

2. BASIC EQUATIONS

Considering gravitational waves in an immobile liquid. Introducing gravity acceleration $\vec{g} = -x_0\, g\ (g = const.)$ in the R. H. S. of the equation of motion:

$$d\vec{V}/dt = -\frac{1}{\rho}\nabla p + \vec{g} \qquad (1)$$

Let the level $x = 0$ correspond to the unperturbed surface of the liquid. When it becomes wavy, the perturbation of the pressure on the level $x = 0$ is, in the linear approximation

$$p_1(0) = \rho\, g\, x_1(0),$$

where $x_1(0) = iVx_1(0)/\omega = -(k/\omega)\psi_1(0)$ is the vertical displacement of the liquid surface in the oscillations. On the other hand, from the y component of the linearized equation of motion then get $p_1 = -(\omega/k)\,\rho\psi_1'$. Consequently, the following relation should hold at $x = 0$;

$$\psi_1'(0) = -g(k/\omega)\psi_1(0) = 0 \qquad (2)$$

In the half space $(x < 0)$ filled by the liquid the perturbation of the stream function satisfies the Rayleigh equation, from which found that $V_0(x) = 0$ at $\psi_1 = \exp(|k|x)$. This solution describe a wave localized on the surface. To satisfy the boundary condition (2), the oscillation frequency should be $\omega = (g|k|)^{1/2}$.

Now taking the motion of the liquid into account. In real flows the velocity of liquid relative to the bottom increases in the direction towards the surface. When gravitational (surface) waves are considered it is natural to use a reference frame connected with the surface of the liquid. In this frame the flow velocity is negative and increases in absolute value towards the interior of the liquid. Since interesting in resonance effects, considering oscillations with $k < 0$ and having, just as the liquid , aa negative phase velocity. Assume that the condition $|gk|^{1/2} >> |V_0'| >> |V_0''/k|$ is met. In this case the influence of the liquid motion on the oscillations can be analyzed by successive approximations. Considering Rayleigh equation after multiplying it by ψ_1^* and subtract from product and after that integrate it from one boundary of the flow to the other, in which putting $x_2 = 0,\ x_1 = -\infty$. Using (2), expressing $\psi_1'(0)$ in terms of $\psi_1(0)$, assuming a complex frequency. Using the relation

$$\frac{\operatorname{Im}\omega}{|\omega - kV_0(x)|^2} \rightarrow_{\operatorname{Im}\omega\to 0} \pi\delta(\omega - kV_0(x))$$

As a result obtaining for $\operatorname{Im}\omega$ the expression

$$\operatorname{Im}\omega = -\frac{\pi}{2}\frac{\omega V_0''(x_s)}{k|V_0'(x_s)|}\left|\frac{\psi_1(x_s)}{\psi_1(0)}\right|^2 \qquad (3)$$

Where $\psi_1(x_s)/\psi_1(0) \approx \exp(-|kx_s|)$, x_s is as before the resonance point at which the phase velocity of the gravitational waves $(\omega = |g/k|^{1/2})$ coincides with the flow velocity.

An example of an unstable velocity when the numbers of resonant particles overtaking and lagging the wave are compared, it must be recognized that in the assumed reference frame both the flow velocity and the oscillation velocity are negative $[V_0(x), \omega/k < 0]$. Therefore the overtaking particles are located below the point x_s. With this circumstance taken into account, it is necessary to use for the quantity df_0 / dV_0 indicative of the ratio of the number of overtaking and lagging particles the expression $df_0 / dV_0 = -\mathrm{sgn} V_0 V_0''/(V_0')^3$. (Recall that the liquid is assumed homogeneous, $\rho_0 \equiv const.$). Resonant buildup of gravitational waves in flow of an ideal liquid of finite depth h was considered in [13]. Analysis of the oscillations with not too large values of $k\,(k \le h^{-1})$ has shown that they build up when Froud's number ($Fr = \Delta V_0 /(gh)^{1/2}$, ΔV_0 is the velocity drop in the flow) exceeds a certain value $Fr_{cr} \approx 0.68$. This result is quite natural. Indeed, at low values of k the phase velocity of the oscillations should, from dimensionality considerations, be of the order of $(gh)^{1/2}$. to satisfy the resonance condition, the velocity drop should be not less than $(gh)^{1/2}$. Resonance effects are particularly pronounced at low viscosity of the liquid $(\mathrm{Re} >> 1)$, when the oscillations can be described by the Rayleigh equation supplemented by the Landau rule for by passing the resonance point. (This is exactly how the instability of gravitational waves was considered in [13]). For low Reynolds numbers and large wavelengths alond $0y$ the resonant layer spreads out over the entire flow $\delta x_s / h \sim (kh\,\mathrm{Re})^{-1/3}$. This case was analyzed in [14,15].

Considering now the buildup of gravitational waves by wind [3,16]. Assume that the air moves above the water surface, i. e., in the region $x > 0$, along $0y$ with velocity $V_0(x)$. The water is assumed to be at rest. For slow oscillations with phase velocity much lower than the sound velocity neglecting the compressibility of the air. Assuming also the air to be homogeneous, weightless, and nonviscous $(\mathrm{Re} >> 1)$, describing these oscillations by the Rayleigh equation.

Establishing the conditions for joining the solutions on the water-air interface. Since the displacements of the air and water particles on the interface are equal, having $\psi_{1+}(0) = \psi_{1-}(0) = \psi_1(0)$; here and below the "+" and "–" signs label quantities pertaining to the state of the air and of the water, respectively. From the y component of the equation of motion of the air then

$$p_{1+}(0) = \rho_+ \{[-(\omega / k) + V_0(0)]\psi_{1+}' - V_0'(0)\psi_1(0)\}$$

On the other side of the interface, the pressure perturbation is

$$p_{1-}(0) = p_{1+}(0) - \rho_- g (k/\omega)\psi_1(0).$$

Recognizing that

$$p_{1-}(0) = -\rho_- g(\omega / k)\psi_{1-}'(0) = -\omega\rho_- \psi_1(0)$$

in a liquid, obtaining ultimately

$$\phi_{1+}'(0) = k\left(\frac{\rho_=}{\rho_+}\left(1 - \frac{gk}{\omega^2}\right)\frac{\omega}{\omega - kV_0(0)} + \frac{V_0'(0)}{\omega - kV_0(0)}\right)\phi_1(0)$$

Since $\rho_- >> \rho_+$, neglecting in the above relation the second in the parentheses.

The frequency of the gravitational waves is $\omega = (gk)^{1/2}$. The resonant interaction with the wind can introduce into the frequency a small imaginary part. To determine $\mathrm{Im}\,\omega$ putting $x_1 = 0,\ x_2 = \infty$. Proceeding next as in the first part of the present section obtaining [16]:

$$\mathrm{Im}\,\omega = -\frac{\rho_+}{\rho_-}\frac{\pi}{2}\frac{\omega}{k}\frac{V_0''(x_s)}{|V_0'(x_s)|^2}\left|\frac{\phi_1(x_s)}{\phi_1(0)}\right|^2 \qquad (4)$$

The above expression differs from (3) only by the factor ρ_+ / ρ_- which is indicative of the lower efficiency of stirring up the heavy liquid by the light air.

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