Mixed Integral Equation of Contact Problem in Position and Time

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***Abstract*--** **In this work we consider a mixed integral equation of the first kind of type Fredholm-Volterra in position and time, respectively. The Fredholm integral term is considered in a variable position, in the space $L_2[-1,1]$, and has a singular kernel. While the Volterra integral term is considered in time, in the class $C[0,T], \ T \prec 1$. Using a numerical method we have a finite system of Fredholm integral equations of the second kind which will be solved numerically, using Chebyshev polynomials method. Numerical results are computed and the error estimate is calculated**

***Index Term*--** **Mixed singular integral equation (MSIE), linear algebraic system (LAS), Chebyshev polynomials (CPs), Fredholm-Volterra integral equation (F-VIE)**
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1. INTRODUCTION

Many problems in mathematical physics, contact problems in the theory of elasticity and mixed boundary value problems lead to integral equations of linear or nonlinear case, see [1-3]. These integral equations with continuous or discontinuous kernels have received considerable interest in mathematical applications in different area of sciences, for example see [4-7]. The solution of these problems can be obtained analytically, see [8-11]. At the same time the sense of numerical methods takes an important place in solving the integral equations, see [3, 12- 13]. The solution of the **MSIE** of type **F-VIE** in position and time in one, two and three dimensions was discussed in [9]. Also, in [9], an asymptotic numerical method was used to discuss, the solution of **F-VIE** of the second kind. The same author, in [10], applied the regular and singular asymptotic method in one, two, and three dimensional, to obtain the solution of **F-VIE** of the first kind.

Consider the **MSIE**

$$\int_0^t \int_{-1}^1 k\left(\left|\frac{y-x}{\lambda}\right|\right) F(t,\tau)\varphi(y,\tau)dy\, d\tau + \int_0^t G(t,\tau)\varphi(x,t)d\tau = \pi\theta[\gamma(t) - f_r(x)] = f(x,t),$$

$$(|x| \prec 1, \ \ \lambda \in (0,\infty), \ \ \theta = G(1-\nu)^{-1}, \ f_r(x) \in L_2[-1,1]), \tag{1.1}$$

$$k(z) = \int_0^\infty \frac{L(u)\cos u\ z}{u} du, \qquad L(u) = \frac{u+m}{1+u}, \ \ z = \frac{y-x}{\lambda}, \ \ (m \geq 1). \tag{1.2}$$

Under the condition

$$\int_{-1}^1 \phi(x,t)dx = P(t), \ \ t \in [0,T], \ \ T \prec 1. \tag{1.3}$$

The **MSIE** (1.1) with its badly kernel of position (1.2) and the two continuous kernels of time F(t, τ); G(t, τ) under the pressure condition (1.3), is investigated from the contact problem of an elastic material of a strip (G_1, ν_1) of thickness h that occupies a region $0 \leq y \leq h$, and lies without friction on an elastic surface (G_2, ν_2) of equation $f_r(x) \in L_2[-1,1]$. Here $G_i; \nu_i$, $i = 1,2$ are called the displacement magnitude and Poisson's ratio for the upper and lower surface respectively.

Consider a rigid rectangular stamp of length 2a is impressed into the boundary of the strip $y = h$ by a variable force $P(t),\ t \in [0,T]$. This variable force causes displacement $\gamma(t)$ against the force of material of the contact region $F(t,\tau)$. Also, consider the contact region has the resistance force $G(t,\tau)$ for all $t,\tau \in [0,T],\ T \prec 1$.

In order to guarantee the existence of a unique solution of equation (1.1), under the pressure condition (1.3), we assume the following conditions

(i) The kernel of the position $k\left(\left|\frac{x-y}{\lambda}\right|\right)$ satisfies, in the space $L_2[-1,1]$ Fredholm condition $\left\{\int_{-1}^{1}\int_{-1}^{1} k^2\left(\left|\frac{y-x}{\lambda}\right|\right)dxdy\right\}^{1/2} = A,\quad A$ is a constant.

(ii) The two kernels of time $F(t,\tau)$ and $G(t,\tau)$ for $t,\ \tau \in [0,T],\ T \prec 1$ belong to the class $C[0,T]$ and satisfies $\left|F(t,\tau)\right| \le B,\ \ \left|G(t,\tau)\right| \le D,$ $B,\ D$ are constants.

(iii) The given function f(x, t), free term function, is continuous with its partial derivatives in the space $L_2[-1,1]\times C[0,T]$ and its norm is defined as: $\|f\| = \max_{0\le t\le T}\left\{\int_{-1}^{1} f^2(x,t)dx\right\}^{1/2}$

(iv) The unknown function $\phi(x,t)$, potential function, satisfies Lipschitz condition with respect to its first argument and Hölder to the second argument.

In this work, we consider, under the conditions (i-iv) the existence and uniqueness solution of a **MSIE** (1.1). The **MSIE** of the first kind (1.1) is considered in position and time. The Fredholm integral term is considered in position, in $L_2[-1,1]$, and its kernel has a singular term and will be adapted to take a logarithmic form. While, the Volterra integral term is considered in time in the space $C[0,T],\ T \prec 1,$ and its kernels are continuous functions with its derivatives. Using a suitable numerical method with respect to time, the mixed integral equation is reduced to an **ALS** of **FIEs** of the second kind. Then using **CPs** the solution of the **FIEs** can be obtained. Numerical results are computed and the error estimate is calculated.

2. THE KERNEL OF POSITION

The function $L(u)$ of equation (1.2) is continuous and positive, for $u \in (0,\infty)$, and then it satisfies the following asymptotic equalities:

$$L(u) = m-(m-1)u+O(u^3),\ \ u\to 0,$$

$$L(u) = 1-\frac{m-1}{u}+O(u^{-2}),\quad u\to\infty,\quad m\ge 1. \tag{2.1}$$

When $m=1$ in (2.1) and $\lambda\to\infty$ in (1.1), such that the term $(\frac{y-x}{\lambda})$ is very small, we have from [6] that

$$\int_0^\infty \frac{\cos uz}{u}du = -\ln|x-y|+d\ ,\qquad \left(d=\ln\frac{4\lambda}{\pi}\right). \tag{2.2}$$

In this case, the kernel of position takes a logarithmic function form.

Assume, in (2.1), $u \to 0$. Then, consider the first and the second approximation of $L(u)$ and the following famous relations [6]

$$\frac{1}{\pi}\int_0^\infty \cos vx dv = \delta(x), \quad \delta(x) \text{ is the Durak function,}$$

$$\int_a^b \phi(y)\delta(y-x)dy = \begin{cases} 0, & b \prec x \prec a, \\ \tfrac{1}{2}[h(x-a)+h(x+a)], & a \prec x \prec b. \end{cases} \quad (2.3)$$

Hence, the **MSIE** (1.1) will be reduced to the following integral equation:

$$\int_0^t H(t,\tau)\phi(x,\tau)\,d\tau - \lambda\int_0^t\int_{-1}^1 F(t,\tau)\ln|x-y|\phi(y,\tau)\,dy\,d\tau = g(x,t)\,, \quad (2.4)$$

where

$$H(t,\tau) = \frac{G(t,\tau)}{m-1} + F(t,\tau), \qquad (m \succ 1), \quad (2.5)$$

and

$$g(x,t) = \frac{f(x,t)}{m-1} + \lambda d\int_0^t P(\tau)F(t,\tau)d\tau,$$

$$\left(d = \ln\frac{4\lambda}{\pi}, \quad \lambda = \frac{m}{m-1}, \quad m \succ 1\right). \quad (2.6)$$

From the three formulas (2.4) – (2.6) the importance of the physical meaning between λ, m is:

1- The Fredholm condition for the logarithmic kernel leads to

$$\left\{\int_{-1}^1\int_{-1}^1 \ln^2|x-y|dx\,dy\right\}^{½} \prec \frac{1}{\lambda}, \qquad \lambda = 1+\frac{1}{m}+\frac{1}{m^2}+..... \quad (2.7)$$

2- Also, from (2.5), we can establish that: for large values of m the total resistance force $H(t,\tau)$ will be equal to the resistance force F (t, τ), i.e. $m \to \infty$ is not a available and the total resistance, in this case, is the resistance force of material only.

3- Also**,** the given function of the free term of (2.4) will depend, in the case, in the value of the pressure P(t) and the resistance force of material.

3. System of Fredholm Integral Equations

To obtain the solution of (2.4), under the condition (1.3), we divide the interval [0,t] as $0 = t_o \prec t_1 \prec t_2 \prec ... \prec t_N = T$, let $t = t_i$, $0 \le i \le N$. Then we approximate the Volterra integral terms, after using the quadrature formula w_j, $j = 0,1,2,...,N$, see [12,13] to have

$$\sum_{j=0}^{i}\omega_j H_{i,j}\,\phi_j(x) - \lambda\sum_{j=0}^{i}\omega_j F_{i,j}\int_{-1}^{1}\ln|x-y|\phi_j(y)dy + O(\hbar_i^{\ell+1}) = g_i(x). \quad (3.1)$$

Under the condition

$$\int_{-1}^1 \phi_i(x)dx = P_i, \qquad (0 \le i \le N). \quad (3.2)$$

Here, $\hbar_i = \max_{0\le j\le N} h_j$, $h_j = t_{j+1} - t_j$ *and* ω_j are called the characteristic points and ℓ is called the quadratic coefficients. The values of ω_j and ℓ depend on the number of derivatives of $F(t,\tau)$ and $G(t,\tau)$.

More information for the characteristic points and coefficients are found in [12, 13]. The formula (3.1) can be adapted in the following form:

$$\mu_i\,\phi_i\,(x) - \ell_i \int_{-1}^{1} \ln|x-y|\,\phi_i\,(y)\,dy = m_i\,(x), \tag{3.3}$$

where

$$m_i(x) = g_i(x) + \sum_{j=0}^{i-1} \omega_i H_{i,j} \phi_j(x) + \lambda \sum_{j=0}^{i-1} \omega_j F_{i,j} \int_{-1}^{1} \ln|x-y|\,\phi_j(y)\,dy,$$

and

$$\mu_i = \omega_i H_{i,i} \;\;,\;\; \ell_i = \lambda \omega_i F_{i,i}. \tag{3.4}$$

The formula (3.3) represents **LAS** of **FIEs** of the second kind with logarithmic kernel.

As an important special case from (3.3) when $i = 0$, we obtain

$$\mu_o \phi_o\,(x) - \ell_o \int_{-1}^{1} \ln|x-y|\,\phi_o\,dy = g_o\,(x). \tag{3.5}$$

Differentiating (3.5) with respect to x, we have

$$\mu_o \frac{d\phi_o(x)}{dx} - \ell_o \int_{-1}^{1} \frac{\phi_o(y)}{x-y} dy = h(x), \quad h(x) = \frac{dg_o(x)}{dx} \tag{3.6}$$

Here, in (3.6), $\int_{-1}^{1}$ denotes integration in the sense of Cauchy principal value and the unknown function $\phi_o(x)$ with its derivatives are continuous in $L_2[-1,1]$, $x \in [-1,1]$. Taking the transformations $y = 2u-1,\; x = 2v-1,$ the integro differential (3.6), on noting the difference notations, becomes

$$\frac{d\Theta}{dv} - \tilde{\lambda} \int_0^1 \frac{\Theta(u)}{v-u} du = z\,(v). \tag{3.7}$$

This equation has appeared in both combined infrared gaseous radiations and molecular conduction, where $\tilde{\lambda}$, in (3.7), is known as the radiation conduction number for the large path length limit, and represents the single parameter of the dimensionless system. The formula (3.7) is considered and discussed with its special cases and solved, when $z\,(v) = \frac{1}{2} - v$, under the conditions $\Theta(0) = \Theta(1) = 0$ by Frankel in [5], where Θ represents the unknown temperature.

4. CHEBYSHEV POLYNOMIALS

To obtain the solution of (3.3), we use the **CPs** with its famous relations. For this, we write the unknown functions $\phi_i\,(x)$, for each value i, $0 \le i \le N$, in the form of the weight function of **CPs** of the first kind $(1-x^2)^{\frac{-1}{2}}$ multiplying by unknown functions $B_i\,(x),\;\; 0 \le i \le N$. Then we write $B_i\,(x)$ in the **CPs** to get

$$\phi_i(x) = \frac{1}{\sqrt{1-x^2}} \sum_{n=0}^{\infty} a_n^{(i)} T_n(x). \tag{4.1}$$

Here in (4.1), the function $T_n(x)$ is called **CP** of the first kind and order n and $a_n^{(i)}$ are the unknown coefficients of $T_n(x)$, will be determined. It is difficult to obtain the solution of equation (3.3) numerically in the form of equation (4.1). For this, the formula (4.1) must be truncated to the following:

$$\phi_i^{(M)} = \frac{1}{\sqrt{1-x^2}} \sum_{n=0}^{M} a_n^{(i)} T_n(x). \tag{4.2}$$

Using (4.2) and the following famous relationships [10],

$$\int_{-1}^{1} \frac{\ln|x-y| T_n(y) dy}{\sqrt{1-y^2}} = \begin{cases} \pi \ln 2, & n=0 \\ \frac{\pi}{n} T_n(x), & n \geq 1, \end{cases} \tag{4.3}$$

the formula (3.3) yields

$$\mu_i \sum_{n=0}^{M} \frac{a_n^{(i)} T_n(x)}{\sqrt{1-x^2}} - \ell_i \begin{cases} \pi\, a_0^{(i)} \ln 2, & n=0 \\ \pi \sum_{n=1}^{M} \frac{a_n^{(i)} T_n(x)}{n}, & n \geq 1 \end{cases}$$

$$= \sum_{n=0}^{M} \frac{g_n^{(i)} T_n(x)}{\sqrt{1-x^2}} + \sum_{j=0}^{i-1} \sum_{n=0}^{M} \omega_j H_{i,j} \frac{a_n^{(i)} T_n(x)}{\sqrt{1-x^2}} - \lambda \sum_{j=0}^{i-1} \omega_j F_{i,j} \begin{cases} \pi \ln 2\, a_0^{(i)}, & n=0 \\ \pi \sum_{n=1}^{M} \frac{a_n^{(i)} T_n(x)}{n}, & n \geq 1 \end{cases} \tag{4.4}$$

where

$$g_n^{(i)} = \frac{2}{\pi} \int_{-1}^{1} \frac{g_{(i)}(x) T_n(x)}{\sqrt{1-x^2}} dx . \tag{4.5}$$

The formula (4.4) leads us to discuss the following two cases:

Case 1: at $n = 0$ we have

$$\mu_i \frac{a_0^{(i)}}{\sqrt{1-x^2}} - \pi \ell_i a_0^{(i)} \ln 2 = \frac{g_0^{(i)}}{\sqrt{1-x^2}} + \sum_{j=0}^{i-1} \omega_j H_{i,j} \frac{a_0^{(j)}}{\sqrt{1-x^2}} - \lambda \pi \ln 2 \sum_{j=0}^{i-1} \omega_j F_{i,j} a_0^{(j)} . \tag{4.6}$$

Integrating (4.6) with respect to x from -1 to 1 we get

$$a_0^{(i)} = \frac{1}{(\mu_i - 2\, \ell_i \ln 2)} [2 g_0^{(i)} + \sum_{j=0}^{i-1} \omega_j H_{i,j}\, a_0^{(j)} + 2 \ln 2\, \lambda \sum_{j=0}^{i-1} \omega_j F_{i,j} a_0^{(j)}],$$

$$(\mu_i \neq 2\, \ell_i \ln 2, \quad 0 \leq i \leq N). \tag{4.7}$$

Case 2: For $n \geq 1$, the formula (4.4) yields

$$\mu_i \sum_{n=1}^{M} \frac{a_n^{(i)} T_n(x)}{\sqrt{1-x^2}} - \pi \ell_i \sum_{n=1}^{M} \frac{a_n^{(i)} T_n(x)}{n} = \sum_{n=1}^{M} \frac{g_n^{(i)} T_n(x)}{\sqrt{1-x^2}} + \sum_{j=0}^{i-1} \sum_{n=1}^{M} \omega_j H_{i,j} \frac{a_n^{(j)} T_n(x)}{\sqrt{1-x^2}}$$
$$+\pi\lambda \sum_{j=0}^{i-1} \sum_{n=1}^{M} \omega_j F_{i,j} \frac{a_n^{(j)} T_n(x)}{n}. \tag{4.8}$$

Multiplying both sides of (4.8) by the term $T_m(x)dx$ and integrating from $x=-1$ to $x=1$, then using the following famous relations [14]

$$T_n(x) T_m(x) = \tfrac{1}{2}\left[T_{n+m}(x) + T_{|n-m|}(x)\right],$$

$$\int_{-1}^{1} T_n(x)dx = \begin{cases} 0 & , n = 1,3,5,\ldots \\ \dfrac{2}{1-n^2} & , n = 0,2,4,\ldots \end{cases} \tag{4.9}$$

we obtain the following **LAS**:

$$\mu_i a_m^{(i)} - 2\ell_i \sum_{n=1}^{M} \frac{A_{n,m}}{n} a_n^{(i)} = C_m^{(i)}, \qquad (m \geq 1, \;\; 0 \leq i \leq N), \tag{4.10}$$

where

$$A_{n,m} = \begin{cases} \dfrac{1}{1-(n+m)^2} + \dfrac{1}{1-(n-m)^2} \;, & (n+m)\; even \\ 0 & (n+m)\; odd, \end{cases} \tag{4.11}$$

and

$$C_m^{(i)} = g_m^{(i)} + \sum_{j=0}^{i-1} \sum_{n=1}^{M} \omega_j H_{i,j} a_n^{(j)} + 2\lambda \sum_{j=0}^{i-1} \sum_{n=1}^{M} \omega_j F_{i,j} \frac{A_{n,m}}{n} a_n^{(j)}. \tag{4.12}$$

Theorem (4.1): For $M \to \infty$, the infinite **LAS** of (4.10) are bounded and have a unique solution.

Proof: Consider the metric space of real bounded set k is defined as

$$\rho(x_1, x_2) = \sup_{\ell} \left| x_\ell^{(1)} - x_\ell^{(2)} \right|, \qquad x^{(p)} = \{x_\ell^{(p)}\}_{\ell=1}^{\infty} \quad (p = 1,2). \tag{4.13}$$

And an operator $K \subset \{k\}$ such that

$$y = Kx\,, \;\; y = \{y_\ell\}_{\ell=1}^{\infty}, \qquad x = \{x_\ell\}_{\ell=1}^{\infty}. \tag{4.14}$$

Also, for $C=\{C_\ell\}_{\ell=1}^{\infty}\in k$, we assume the bounded and continuous the following infinite LAS, see [15]

$$y_\ell = C_\ell + \gamma \sum_{n=1}^{\infty} K_{n,\ell} x_n \;, \qquad \gamma \text{ is a constant.} \tag{4.15}$$

Hence, under the condition $\sup_{n,\ell}\left|K_{n,\ell}\right| \prec \infty$, the operator K satisfies $K : k \to k$, i.e. the system (4.15) has a unique solution.

So, in the same way, when $M \to \infty$ we rewrite (4.10) to be

$$\begin{gathered} a_m^{(i)} - \tilde{\lambda}_i \sum_{n=1}^{\infty} R_{n,m} a_n^{(i)} = L_m^{(i)}, \\ \left(\tilde{\lambda}_i = \frac{2\ell_i}{\mu_i}, \;\; L_m^{(i)} = C_m^{(i)} \mu_i^{(-1)}, \quad R_{n,m} = \frac{A_{n,m}}{n} \right). \end{gathered} \tag{4.16}$$

The convergence of the **LAS** of equation (4.16) can be obtained after applying Cauchy – Minkoviski inequality. Therefore, we follow

$$S_m = \tilde{\lambda}_i \sum_{n=1}^{\infty} \left| R_{n,m} \right| = \tilde{\lambda}_i \sum_{n=1}^{\infty} \frac{1}{n} \left| A_{n,m} \right| . \tag{4.17}$$

Hence, we get

$$S_m = \tilde{\lambda}_i \left| \sum_{n=1}^{\infty} \frac{1}{n^2} \right|^{\frac{1}{2}} \left| \sum_{n=1}^{\infty} (A_{n,m})^2 \right|^{\frac{1}{2}} \le 1. \tag{4.18}$$

Using the values of the convergence series $[\sum_{n=1}^{\infty}(A_{n,m})^2] \to 1,\ as\ m \to \infty$, we obtain $\ell_i \le 0.39\,\mu_i$ for all values of i, $0 \le i \le N$.

5. NUMERICAL RESULTS

For the analytical solution of equation (3.4) $\phi(x,t) = x^2 t$, we assume

$$m=2, \quad \lambda = 2, \quad F(t,\tau) = \tau^2, \quad G(t,\tau) = \tau^3; \; k(x-y) = \ln|y-x|, \tag{5.1}$$

Hence, we have

$$H(t,\tau) = \tau^3 - \tau^2, \tag{5.2}$$

and

$$g(x,t)=\frac{1}{5}t^5-\frac{1}{4}t^4\left(x^2+I(x)\right), \qquad (5.3)$$

where

$$I(x)=\frac{1}{3}\left(1-x^3\right)\ln|1-x|+\frac{1}{3}\left(1+x^3\right)ln|1+x|-\frac{2}{9}-\frac{2}{3}x^2.$$

The exact solution and the corresponding numerical solution of the **LAS** (4.10) for the previous data are obtained for the times t=0.1, t=0.4 and t=0, through the following table:

Table(1.1)

t	x	ϕ_E	ϕ_N	E_T
t=.0.1	-0.98	0.0960400	0.0960400	98E-11
	-0.44	0.0193600	0.0193600	67E-11
	0.1	0.0010000	0.0010000	45E-11
	0.64	0.0496000	0.0496000	76E-11
	0.96	0.8836000	0.8836000	95E-11
t=0.4	-0.98	0.3841600	0.3841608	8.6E-9
	-0.44	0.07744000	0.07744005	5.6E-9
	0.1	0.00400000	0.00400002	2.7E-9
	0.64	0.16384000	0.16384006	6.4E-9
	0.96	0.36864000	0.36864008	8.1E-9
t=0.9	-0.98	0.86436000	0.8643653	5.3E-6
	-0.44	0.17424000	0.1742478	7.8E-6
	0.1	0.0090000	0.0090047	4.7E-6
	0.64	0.3686400	0.3686434	3.4E-6
	0.96	0.8294400	0.8294451	5.1E-6

We note that from the above table and other numerical results that:

1- As the time increases the error decreases. Also, as m increases the error decreases.

2- The analytical solution and the numerical one is very small which assure the accuracy of the numerical and the analytical techniques considered in this paper.

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