Perturbation Solution of Fourth Order Critically Damped Oscillatory Nonlinear Systems

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Abstract

A perturbation technique is developed in this article based on the KBM method to investigate the solution of fourth order critically damped oscillatory nonlinear systems in the case of two roots are real and equal and the other two roots are complex conjugate. The results obtained by this method show excellent coincidence with those obtained by numerical method. . The method is illustrated by an example.

1. Introduction

The Krylov-Bogoliubov-Mitropolskii (KBM) method [4], [6] is well known in the theory of nonlinear oscillations. The method was originally developed by Krylov and Bogoliubov [6] for obtaining the periodic solution of non linear systems with small nonlinearities. Now the method is used to obtain the solutions of oscillatory, damped oscillatory, critically damped, more critically damped and non-oscillatory systems with second, third, fourth etc. order non linear differential equation by imposing some restrictions to make the solution uniformly valid. The method was extended by Popov [10] for nonlinear damped oscillatory systems. Murty, Deekshatulu, and Krisna [7] investigated an over damped nonlinear system using Bogoliubov's method. Murty [9] presented a unified KBM method for solving second order nonlinear systems which cover the un-damped, damped and over-damped cases. Shamsul and Sattar [14] also presented a unified method for obtaining approximate solutions of third order damped and over-damped oscillatory nonlinear systems based on the KBM method. Later, Alam [16] extended the method for n-th order nonlinear systems. Akbar et al. [1] investigated a technique for solving fourth order over-damped nonlinear systems. Akbar et al. [2] extended the technique for damped oscillatory nonlinear systems in the case when the four eigen values are complex conjugate. Later, Habibur et al [5] presented a technique for fourth order damped oscillatory nonlinear systems in the case when two of the eigen values are real and distinct and the other two are complex conjugate. But none of the above authors investigate the solution of fourth order nonlinear systems when two of the eigen values are real and equal and the other two are complex conjugate.

We have extended the KBM method in this article to investigate the solution of fourth order critically damped oscillatory nonlinear systems when two of the eigen values are real and equal and the other two are complex conjugate. The solutions obtained by the presented method show nice coincidence with those obtained by numerical method.

2. The method

Let us consider the following fourth order weakly nonlinear differential equation

$$x^{(4)} + c_1 \dddot{x} + c_2 \ddot{x} + c_3 \dot{x} + c_4 x = -\varepsilon\, f(x), \quad (1)$$

where $x^{(4)}$ stands for fourth derivative of x with respect to t, over dots are used for the first, second and third derivatives, ε is the small parameter, c_1, c_2, c_3, c_4 are constants, and f is the given nonlinear function. Since the system is critically damped oscillatory, so two of the eigenvalues are real and equal and the other two are complex conjugate. Suppose two eigen values say $-\lambda_1$, $-\lambda_2$ are real and equal and the other two $-\lambda_3$ and $-\lambda_4$ are complex conjugate.

When $\varepsilon = 0$, the unperturbed solution of the equation (1) is

$$x(t,0) = \left(a_{1,0} + t\, a_{2,0}\right)\, e^{-\lambda_2 t} + a_{3,\,0}\, e^{-\lambda_3 t} + a_{4,\,0}\, e^{-\lambda_4 t}, \quad (2)$$

where $a_{j,\,0}$ $(j = 1, 2, 3, 4)$ are constants of integration.

When $\varepsilon \neq 0$, following Shamsul [13] a solution of the equation (1) is sought in the form

$$x(t,\varepsilon) = (a_1(t) + t\ a_2(t))e^{-\lambda_2 t} + a_3(t)\ e^{-\lambda_3 t} + a_4(t)\ e^{-\lambda_4 t} + \varepsilon u_1(a_1,a_2,a_3,a_4,t) + \ldots, \quad (3)$$

where each a_j, $(j = 1, 2, 3, 4)$ satisfy the first order differential equation

$$\dot{a}_j(t) = \varepsilon A_j(a_1,a_2,a_3,a_4,t) + \ldots \quad (4)$$

Now differentiating the equation (3) four times with respect to t, substituting the value of x and the derivatives $\dot{x}, \ddot{x}, \dddot{x}, x^{(4)}$ in the equation (1), utilizing the relation presented in (4) and finally equating the coefficients of ε, we obtain

$$e^{-\lambda_2 t}\left(\frac{\partial}{\partial t} - \lambda_2 + \lambda_3\right)\left(\frac{\partial}{\partial t} - \lambda_2 + \lambda_4\right)\left(\frac{\partial A_1}{\partial t} + 2A_2 + t\frac{\partial A_2}{\partial t}\right) + e^{-\lambda_3 t}\left(\frac{\partial}{\partial t} - \lambda_3 + \lambda_2\right)^2\left(\frac{\partial}{\partial t} - \lambda_3 + \lambda_4\right)A_3 + e^{-\lambda_4 t}\left(\frac{\partial}{\partial t} - \lambda_4 + \lambda_2\right)^2\left(\frac{\partial}{\partial t} - \lambda_4 + \lambda_3\right)A_4 + \left(\frac{\partial}{\partial t} + \lambda_2\right)^2\left(\frac{\partial}{\partial t} + \lambda_3\right)\left(\frac{\partial}{\partial t} + \lambda_4\right)u_1 = -f^{(0)}(a_1,a_2,a_3,a_4,t), \quad (5)$$

where $f^{(0)}(a_1,a_2,a_3,a_4,t) = f(x_0)$ and $x_0 = (a_1 + a_2\,t)e^{-\lambda_2 t} + a_3\,e^{-\lambda_3 t} + a_4\,e^{-\lambda_4 t}$.

In this article, we have assumed that the functional $f^{(0)}$ can be expanded in a Taylor's series (see also Murty and Deekshatulu [7], Sattar [12], Shamsul and Sattar [13]) in the form

$$f^{(0)} = \sum_{j,k,l,m=0}^{\infty} F_{0,m}(a_1,a_2,a_3,a_4)\, e^{-(j\lambda_2 + k\lambda_3 + l\lambda_4)t} + (a_1 + a_2\,t)\sum_{j,k,l,m=0}^{\infty} F_{1,m}(a_1,a_2,a_3,a_4)\, e^{-(j\lambda_2 + k\lambda_3 + l\lambda_4)t} + (a_1 + a_2\,t)^2 \sum_{j,k,l,m=0}^{\infty} F_{2,m}(a_1,a_2,a_3,a_4)\, e^{-(j\lambda_2 + k\lambda_3 + l\lambda_4)t} + (a_1 + a_2\,t)^3 \sum_{j,k,l,m=0}^{\infty} F_{3,m}(a_1,a_2,a_3,a_4)\, e^{-(j\lambda_2 + k\lambda_3 + l\lambda_4)t} + \ldots \quad (6)$$

Substituting the value of $f^{(0)}$ from equation (6) into equation (5), we obtain

$$e^{-\lambda_2 t}\left(\frac{\partial}{\partial t} - \lambda_2 + \lambda_3\right)\left(\frac{\partial}{\partial t} - \lambda_2 + \lambda_4\right)\left(\frac{\partial A_1}{\partial t} + 2A_2 + t\frac{\partial A_2}{\partial t}\right) + e^{-\lambda_3 t}\left(\frac{\partial}{\partial t} - \lambda_3 + \lambda_2\right)^2\left(\frac{\partial}{\partial t} - \lambda_3 + \lambda_4\right)A_3 + e^{-\lambda_4 t}\left(\frac{\partial}{\partial t} - \lambda_4 + \lambda_2\right)^2\left(\frac{\partial}{\partial t} - \lambda_4 + \lambda_3\right)A_4 + \left(\frac{\partial}{\partial t} + \lambda_2\right)^2\left(\frac{\partial}{\partial t} + \lambda_3\right)\left(\frac{\partial}{\partial t} + \lambda_4\right)u_1 = -\sum_{j,k,l,m=0}^{\infty} F_{0,m}(a_1,a_2,a_3,a_4)\, e^{-(j\lambda_2 + k\lambda_3 + l\lambda_4)t} - (a_1 + a_2\,t)\sum_{j,k,l,m=0}^{\infty} F_{1,m}(a_1,a_2,a_3,a_4)\, e^{-(j\lambda_2 + k\lambda_3 + l\lambda_4)t} - (a_1 + a_2\,t)^2 \sum_{j,k,l,m=0}^{\infty} F_{2,m}(a_1,a_2,a_3,a_4)\, e^{-(j\lambda_2 + k\lambda_3 + l\lambda_4)t} - (a_1 + a_2\,t)^3 \sum_{j,k,l,m=0}^{\infty} F_{3,m}(a_1,a_2,a_3,a_4)\, e^{-(j\lambda_2 + k\lambda_3 + l\lambda_4)t} + \ldots \quad (7)$$

KBM [4], [6], Sattar [12], Shamsul [15], [17], [20], Shamsul and Sattar [13] imposed the condition that u_1 cannot contain the fundamental terms (the solution presented in equation (2) is called generating solution and its terms are called fundamental terms) of $f^{(0)}$. *i. e.*, the terms $(a_1 + t\ a_2)^0$ and $(a_1 + t\ a_2)^1$. Therefore, equation (7) can be separated for the unknown functions u_1 and A_1, A_2, A_3, A_4 in the following way:

$$e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_4\right)\left(\frac{\partial A_1}{\partial t}+2A_2+t\frac{\partial A_2}{\partial t}\right)+e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)A_3+e^{-\lambda_4 t}\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_3\right)A_4=-\sum_{j,k,l,m=0}^{\infty}F_{0,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t}-(a_1+a_2 t)\sum_{j,k,l,m=0}^{\infty}F_{1,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t} \tag{8}$$

and

$$\left(\frac{\partial}{\partial t}+\lambda_2\right)^2\left(\frac{\partial}{\partial t}+\lambda_3\right)\left(\frac{\partial}{\partial t}+\lambda_4\right)u_1=-(a_1+a_2 t)^2\sum_{j,k,l,m=0}^{\infty}F_{2,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t}-(a_1+a_2 t)^3\sum_{j,k,l,m=0}^{\infty}F_{3,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t}-\cdots \tag{9}$$

Now, equating the coefficients of t^0 and t^1 from both sides of the equation (8), we obtain

$$e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_4\right)\left(\frac{\partial A_1}{\partial t}+2A_2\right)+e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)A_3+e^{-\lambda_4 t}\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_3\right)A_4=-\sum_{j,k,l,m=0}^{\infty}F_{0,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t}-a_1\sum_{j,k,l,m=0}^{\infty}F_{1,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t}, \tag{10}$$

and

$$e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_4\right)\frac{\partial A_2}{\partial t}=-a_2\sum_{j,k,l,m=0}^{\infty}F_{1,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t} \tag{11}$$

Solving equation (11), we obtain

$$A_2=\sum_{j,k,l,m=0}^{\infty}\frac{a_2F_{1,m}(a_1,a_2,a_3,a_4).e^{-((j-1)\lambda_2+k\lambda_3+l\lambda_4)t}}{((j-1)\lambda_2+k\lambda_3+l\lambda_4).(\lambda_2+(k-1)\lambda_3+l\lambda_4).(j\lambda_2+k\lambda_3+(l-1)\lambda_4)} \tag{12}$$

Substituting the value of A_2 from equation (12) into equation (10), we obtain

$$e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_4\right)\frac{\partial A_1}{\partial t}+e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)A_3+e^{-\lambda_4 t}\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_3\right)A_4=-\sum_{j,k,l,m=0}^{\infty}F_{0,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t}-a_1\sum_{j,k,l,m=0}^{\infty}F_{1,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t}-2e^{-\lambda_2 t}.\sum_{j,k,l,m=0}^{\infty}\frac{a_2F_{1,m}(a_1,a_2,a_3,a_4)e^{-\left((j-1)\lambda_2+k\lambda_3+l\lambda_4\right)t}}{((j-1)\lambda_2+k\lambda_3+\lambda_4)} \tag{13}$$

Now it is not easy to solve the equation (13) for the unknown functions A_1, A_3 and A_4, if the nonlinear function f and the eigen values $-\lambda_2, -\lambda_3, -\lambda_4$ of the linear equation of (1) are not specified. When these are specified the values of A_1, A_3 and A_4 can be found subject to the condition that the coefficient in the solution of A_1, A_3 and A_4 do not become large (see also Ali Akbar *et al*. [3], Shamsul [18], [19], [21] for details). For this reason, we have considered that the relations $\lambda_2 \approx 2\lambda_3+2\lambda_4$ and $\lambda_1=\lambda_2$

exist among the eigenvalues. These relations are important, since under these relations the coefficients in the solution of A_1, A_3 and A_4 do not become large. Substituting the value A_1, A_2, A_3, A_4 into equation (4), we obtain, $\dot{a}_j(t),\ j=1,2,3,4$. Since $\dot{a}_j(t),\ j=1,2,3,4$ are proportional to the small parameter ε, so they are slowly varying functions of time *t*. Hence their rate of change are very small *i. e.*, they are almost constant. Therefore, it is plausible to replace a_1, a_2, a_3, a_4 by their respective values obtained in the linear case (*i. e.* the values of a_1, a_2, a_3, a_4 obtained when $\varepsilon = 0$) in the right hand side of (4). This replacement was first made by Murty *et al.* [8], [9] to solve similar type of nonlinear equations. Thus substituting the values of A_1, A_2, A_3 and A_4 into the equation (4) and integrating, we obtain

$$\left.\begin{aligned} a_1 &= a_{1,0} + \varepsilon \int_0^t A_1(a_{1,0}, a_{2,0}, a_{3,0}, a_{4,0}, t)\, dt \\ a_2 &= a_{2,0} + \varepsilon \int_0^t A_2(a_{1,0}, a_{2,0}, a_{3,0}, a_{4,0}, t)\, dt \\ a_3 &= a_{3,0} + \varepsilon \int_0^t A_3(a_{1,0}, a_{2,0}, a_{3,0}, a_{4,0}, t)\, dt \\ a_4 &= a_{4,0} + \varepsilon \int_0^t A_4(a_{1,0}, a_{2,0}, a_{3,0}, a_{4,0}, t)\, dt \end{aligned}\right\} \quad (14)$$

Substituting the values of a_1, a_2, a_3, a_4 and u_1 in the equation (3), we get the complete solution of (1). Thus the determination of the first order approximate solution is completed.

3. Example

As an example of the above method, we consider the Duffing equation type nonlinear equation

$$x^{(4)} + c_1 \dddot{x} + c_2 \ddot{x} + c_3 \dot{x} + c_4 x = -\varepsilon x^3, \qquad (15)$$

The unperturbed solution of (15) as prescribed in (2) can be rewritten as

$$x(t,0) = \frac{1}{2} a e^{-k_1 t}\left\{e^{-(\omega_1 t-\phi)} + t e^{-(\omega_1 t+\phi)}\right\} + b e^{-k_2 t}\cos(\omega_2 t + \varphi).$$

Also, from (3), the first approximate solution is

$$x(t,\varepsilon) = \frac{1}{2} a e^{-k_1 t}\left\{e^{-(\omega_1 t-\phi)} + t e^{-(\omega_1 t+\phi)}\right\} + b e^{-k_2 t}\cos(\omega_2 t + \varphi) + \varepsilon u_1$$

Here, $f = x^3$ and $x_0 = (a_1 + a_2 t) e^{-\lambda_2 t} + a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t}$.

Therefore,

$$\begin{aligned} f^{(0)} &= a_3^3 e^{-3\lambda_3 t} + 3a_3^2 a_4 e^{-(2\lambda_3+\lambda_4)t} \\ &+ 3a_3 a_4^2 e^{-(\lambda_3+2\lambda_4)t} + a_4^3 e^{-3\lambda_4 t} \\ &+ 3(a_1 + a_2 t)\begin{pmatrix} a_3^2 e^{-2\lambda_3 t} + 2a_3 a_4 e^{-(\lambda_3+\lambda_4)t} \\ + a_4^2 e^{-2\lambda_4 t} \end{pmatrix} \\ &+ 3(a_1 + a_2 t)^2 \left(a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t}\right) \\ &+ e^{-3\lambda_2 t}(a_1 + a_2 t)^3 \end{aligned} \qquad (16)$$

Therefore, comparing equation (6) and (16), we obtain

$$\sum_{j,k,l,m=0}^{\infty} F_{0,m}(a_1, a_2, a_3, a_4) e^{-(j\lambda_2 + k\lambda_3 + l\lambda_4)t} = a_3^3 e^{-3\lambda_3 t} + 3a_3^2 a_4 e^{-(2\lambda_3+\lambda_4)t} + 3a_3 a_4^2 e^{-(\lambda_3+2\lambda_4)t} + a_4^3 e^{-3\lambda_4 t},$$

$$\sum_{j,k,l,m=0}^{\infty} F_{1,m}(a_1, a_2, a_3, a_4) e^{-(j\lambda_2 + k\lambda_3 + l\lambda_4)t} = 3\begin{pmatrix} a_3^2 e^{-2\lambda_3 t} + 2a_3 a_4 e^{-(\lambda_3+\lambda_4)t} \\ + a_4^2 e^{-2\lambda_4 t} \end{pmatrix},$$

$$\sum_{j,k,l,m=0}^{\infty} F_{2,m}(a_1, a_2, a_3, a_4) e^{-(j\lambda_2 + k\lambda_3 + l\lambda_4)t} = 3\left(a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t}\right),$$

and

$$\sum_{j,k,l,m=0}^{\infty} F_{3,m}(a_1, a_2, a_3, a_4) e^{-(j\lambda_2 + k\lambda_3 + l\lambda_4)t} = e^{-3\lambda_2 t}. \qquad (17)$$

Therefore, equations (9)-(11) respectively become

$$\left(\frac{\partial}{\partial t}+\lambda_2\right)^2\left(\frac{\partial}{\partial t}+\lambda_3\right)\left(\frac{\partial}{\partial t}+\lambda_4\right)u_1 = -(a_1+a_2\,t)^2\,3\left(a_3\,e^{-\lambda_3\,t}+a_4\,e^{-\lambda_4\,t}\right) -(a_1+a_2\,t)^3\,e^{-3\lambda_2\,t}, \quad (18)$$

$$e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_4\right)\left(\frac{\partial A_1}{\partial t}+2\,A_2\right)+e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)A_3+e^{-\lambda_4 t}\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_3\right)A_4=-\{a_3^3e^{-3\lambda_3 t}+3a_3^2\,a_4\,e^{-(2\lambda_3+\lambda_4)t}+3a_3\,a_4^2\,e^{-(\lambda_3+2\lambda_4)t}+a_4^3e^{-3\lambda_4 t}\}-3a_1\begin{pmatrix}a_3^2\,e^{-2\lambda_3 t}+2a_3\,a_4\,e^{-(\lambda_3+\lambda_4)t}\\+a_4^2\,e^{-2\lambda_4 t}\end{pmatrix}, \quad (19)$$

and

$$e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_4\right)\frac{\partial A_2}{\partial t}=-3a_2\begin{pmatrix}a_3^2\,e^{-2\lambda_3 t}+2a_3\,a_4\,e^{-(\lambda_3+\lambda_4)t}\\+a_4^2\,e^{-2\lambda_4 t}\end{pmatrix} \quad (20)$$

Therefore, solving equation (20), we obtain

$$A_2=q_1\,a_2\,a_3^2e^{-2\lambda_3 t}+q_2\,a_2\,a_3\,a_4\,e^{-(\lambda_3+\lambda_4)t}+q_3\,a_2\,a_4^2\,e^{-2\lambda_4 t} \quad (21)$$

where,

$q_1=3/2\lambda_3(\lambda_2+\lambda_3)(\lambda_2+2\lambda_3-\lambda_4)$,

$q_2=6/(\lambda_2+\lambda_3)(\lambda_2+\lambda_4)(\lambda_3+\lambda_4)$,

$q_3=3/2\lambda_4(\lambda_2+\lambda_4)(\lambda_2-\lambda_3+2\lambda_4)$.

Considering $\lambda_1=\lambda_2=k_1+\omega_1$, $\lambda_3=k_2-i\omega_2$ and $\lambda_4=k_2+i\omega_2$, and then substituting these values , we obtain

$$q_1=3/2(k_2-i\omega_2)(k_1+k_2+\omega_1-i\omega_2)(k_1+k_2+\omega_1-3i\omega_2),$$

$$q_2=3/\left((k_1+k_2+\omega_1)^2+\omega_2{}^2\right),$$

$$q_3=3/2(k_2+i\omega_2)(k_1+k_2+\omega_1+i\omega_2)(k_1+k_2+\omega_1+3i\omega_2).$$

Now substituting the value of A_2 from equation (21) into equation (19), we obtain

$$e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_4\right)\frac{\partial A_1}{\partial t}+e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)A_3+e^{-\lambda_4 t}\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_3\right)A_4=-\{a_3^3e^{-3\lambda_3 t}+3a_3^2\,a_4\,e^{-(2\lambda_3+\lambda_4)t}+3a_3\,a_4^2\,e^{-(\lambda_3+2\lambda_4)t}+a_4^3e^{-3\lambda_4 t}\}-3a_1\{a_3^2\,e^{-2\lambda_3 t}+2a_3\,a_4\,e^{-(\lambda_3+\lambda_4)t}+a_4^2\,e^{-2\lambda_4 t}\}-2e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_4\right)\{q_1a_2a_3^2e^{-2\lambda_3 t}+q_2\,a_2\,a_3\,a_4\,e^{-(\lambda_3+\lambda_4)t}+q_3\,a_2\,a_4^2\,e^{-2\lambda_4 t}\}. \quad (22)$$

To separate the equation (22) for determining the unknown functions A_1 A_3 and A_4, in this article we consider the important relations $\lambda_2\approx 2\lambda_3+2\lambda_4$ (Ali Akbar *et al.* [3], Shamsul [18], [19], [21], Rokibul *et al.* [11]) exist among the eigen values and $\lambda_1=\lambda_2$ for critical damping .

Under these conditions, we obtain

$$e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_4\right)\frac{\partial A_1}{\partial t}=-\{3a_1\,a_3^2\,e^{-(\lambda_2+2\lambda_3)t}+6a_1\,a_3\,a_4\,e^{-(\lambda_2+\lambda_3+\lambda_4)t}+3a_1a_4^2\,e^{-(\lambda_2+2\lambda_4)t}\}-\left\{\frac{3a_2a_3^2}{\lambda_3}e^{-(2\lambda_3+\lambda_2)t}+\frac{12\,a_2\,a_3\,a_4}{(\lambda_3+\lambda_4)}e^{-(\lambda_2+\lambda_3+\lambda_4)t}+\frac{3a_2\,a_4^2}{\lambda_4}\,e^{-(2\lambda_4+\lambda_2)t}\right\}, \quad (23)$$

$$e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)A_3 = -\{3a_3\,a_4^2\,e^{-(\lambda_3+2\lambda_4)t}+a_3^3e^{-3\lambda_3 t}\}, \tag{24}$$

and

$$e^{-\lambda_4 t}\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_3\right)A_4 = -\left\{3a_3^2\,a_4\,e^{-(2\lambda_3+\lambda_4)t}+a_4^3e^{-3\lambda_4 t}\right\} \tag{25}$$

The particular solutions of equations (23)-(25) respectively become

$$A_1 = p_1\,a_1\,a_3^2\,e^{-2\lambda_3 t} + p_2\,a_1\,a_3\,a_4\,e^{-(\lambda_3+\lambda_4)t} + p_3\,a_1a_4^2\,e^{-2\lambda_4 t} + p_4\,a_2a_3^2e^{-2\lambda_3 t} + p_5\,a_2\,a_3\,a_4\,e^{-(\lambda_3+\lambda_4)t} + p_6\,a_2\,a_4^2\,e^{-2\lambda_4 t}, \tag{26}$$

$$A_3 = r_1\,a_3\,a_4^2\,e^{-2\lambda_4 t}+r_2\,a_3^3\,e^{-2\lambda_3 t}, \tag{27}$$

and

$$A_4 = s_1\,a_3^2\,a_4\,e^{-2\lambda_3 t}+s_2\,a_4^{\,3}\,e^{-2\lambda_4 t}, \tag{28}$$

where

$p_1 = 3/2(k_2-i\omega_2)(k_1+k_2+\omega_1-i\omega_2)(k_1+k_2+\omega_1-3i\omega_2)$,

$p_2 = 3/k_2\left((k_1+k_2+\omega_1)^2+\omega_2^{\,2}\right)$,

$p_3 = 3/2(k_2+i\omega_2)(k_1+k_2+\omega_1+i\omega_2)(k_1+k_2+\omega_1+3i\omega_2)$,

$p_4 = 3/2(k_2-i\omega_2)(k_1+k_2+\omega_1-i\omega_2)(k_1+k_2+\omega_1-3i\omega_2)$,

$p_5 = 6/k_2\left((k_1+k_2+\omega_1)^2+\omega_2^{\,2}\right)$,

$p_6 = 3/2(k_2+i\omega_2)(k_1+k_2+\omega_1+i\omega_2)(k_1+k_2+\omega_1+3i\omega_2)$,

$r_1 = 3/2k_2(k_1-3k_2+\omega_1-i\omega_2)^2$,

$r_2 = 1/\left((k_1-4k_2+\omega_1+4i\omega_2)^2(3k_2-5i\omega_2)\right)$,

$s_1 = 3/2k_2(k_1-3k_2+\omega_1+i\omega_2)^2$,

$s_2 = 1/\left((k_1+4k_2+\omega_1-4i\omega_2)^2(3k_2+5i\omega_2)\right)$

, as $\lambda_1=\lambda_2=k_1+\omega_1$, $\lambda_3=k_2-i\omega_2$ and $\lambda_4=k_2+i\omega_2$.

Here u_1 is a correction term and has also very small contribution in the solution. However it is laborious to solve (7) for u_1. So we neglect the calculation of u_1. Substituting the values of A_1, A_2, A_3 and A_4 from equation (21),(26)-(28) into (4) and neglecting the second and higher powers of ε (since ε is very small), we obtain

$$\dot a_1 = \varepsilon\left[p_1\,a_1\,a_3^2\,e^{-2\lambda_3 t} + p_2\,a_1\,a_3\,a_4\,e^{-(\lambda_3+\lambda_4)t}+p_3\,a_1a_4^2\,e^{-2\lambda_4 t}+p_4\,a_2a_3^2e^{-2\lambda_3 t}+p_5\,a_2\,a_3\,a_4\,e^{-(\lambda_3+\lambda_4)t}+p_6\,a_2\,a_4^2\,e^{-2\lambda_4 t}\right],$$

$$\dot a_2 = \varepsilon\left[q_1\,a_2\,a_3^2e^{-2\lambda_3 t}+q_2\,a_2\,a_3\,a_4\,e^{-(\lambda_3+\lambda_4)t}+q_3\,a_2\,a_4^2\,e^{-2\lambda_4 t}\right],$$

$$\dot a_3 = \varepsilon\left[r_1\,a_3\,a_4^2\,e^{-2\lambda_4 t}+r_2\,a_3^3\,e^{-2\lambda_3 t}\right],$$

$$\dot a_4 = \varepsilon\left[s_1\,a_3^2\,a_4\,e^{-2\lambda_3 t}+s_2\,a_4^{\,3}\,e^{-2\lambda_4 t}\right] \tag{29}$$

.

Now, substituting $a_1=\dfrac{ae^{\phi}}{2}, a_2=\dfrac{ae^{-\phi}}{2}$, $a_3=\dfrac{be^{i\varphi}}{2}$, $a_4=\dfrac{be^{-i\varphi}}{2}$ into the equation (29) and simplifying, we obtain

$$\dot a = \varepsilon\frac{3}{8}ab^2e^{-2k_2 t}\left\{h_1\cos 2(\omega_2 t+\varphi)+h_2\sin 2(\omega_2 t+\varphi)+h_3+h_4e^{-2\phi}\right\},$$

$$\dot b = \varepsilon b^3\left\{(j_1+j_2)e^{-2k_2 t}\cos 2(\omega_2 t+\varphi)+(j_3+j_4)e^{-2k_2 t}\sin 2(\omega_2 t+\varphi)\right\},$$

$$\dot\phi = \varepsilon\frac{3}{8}b^2\left\{g_1e^{-2(k_2 t+\phi)}\cos 2(\omega_2 t+\varphi)+g_2e^{-2(k_2 t+\phi)}\sin 2(\omega_2 t+\varphi)+g_3e^{-2k_2 t}+g_4e^{-2(k_2 t+\phi)}\right\},$$

$$\dot\varphi = \varepsilon b^2\left\{(f_1+f_2)e^{-2k_2 t}\cos 2(\omega_2 t+\varphi)+(f_3+f_4)e^{-2k_2 t}\sin 2(\omega_2 t+\varphi)\right\}, \tag{30}$$

where

$$h_1 = \frac{\left(e^{2\phi}+2\right)\left[\begin{array}{l}\left\{k_2\left(k_1+k_2+\omega_1\right)-\omega_2^{\,2}\right\}.\\ \left(k_1+k_2+\omega_1\right)-3\omega_2^{\,2}.\\ \left(k_1+2k_2+\omega_1\right)\end{array}\right]}{\begin{array}{l}\left(k_2^{\,2}+\omega_2^{\,2}\right)\left\{\left(k_1+k_2+\omega_1\right)^2+\omega_2^{\,2}\right\}.\\ \left\{\left(k_1+k_2+\omega_1\right)^2+9\omega_2^{\,2}\right\}\end{array}},$$

$$h_2 = \frac{-\left(e^{2\phi}+2\right)\left[\begin{array}{l}3\omega_2\left\{\begin{array}{l}k_2\left(k_1+k_2+\omega_1\right)\\ -\omega_2^{\,2}\end{array}\right\}\\ +\omega_2\left(k_1+2k_2+\omega_1\right).\\ \left(k_1+k_2+\omega_1\right)\end{array}\right]}{\begin{array}{l}\left(k_2^{\,2}+\omega_2^{\,2}\right)\left\{\left(k_1+k_2+\omega_1\right)^2+\omega_2^{\,2}\right\}.\\ \left\{\left(k_1+k_2+\omega_1\right)^2+9\omega_2^{\,2}\right\}\end{array}},$$

$$h_3 = \frac{1+k_2}{k_2\left\{\left(k_1+k_2+\omega_1\right)^2+\omega_2^{\,2}\right\}},$$

$$h_4 = \frac{2}{k_2\left\{\left(k_1+k_2+\omega_1\right)^2+\omega_2^{\,2}\right\}},$$

$$j_1 = \frac{\frac{3}{8}\left[\left(k_1-3k_2+\omega_1\right)^2-\omega_2^{\,2}\right]}{k_2\left[\left(k_1-3k_2+\omega_1\right)^2+\omega_2^{\,2}\right]^2},$$

$$j_2 = \frac{\frac{1}{4}\left[\begin{array}{l}3k_2\left\{\left(k_1-4k_2+\omega_1\right)^2-16\omega_2^{\,2}\right\}\\ +40\omega_2^{\,2}\left(k_1-4k_2+\omega_1\right)\end{array}\right]}{\left(9k_2^{\,2}+25\omega_2^{\,2}\right)\left\{\begin{array}{l}\left(k_1-4k_2+\omega_1\right)^2\\ +16\omega_2^{\,2}\end{array}\right\}^2},$$

$$j_3 = \frac{\frac{3}{4}\omega_2\left(k_1-3k_2+\omega_1\right)}{k_2\left[\left(k_1-3k_2+\omega_1\right)^2+\omega_2^{\,2}\right]^2},$$

$$j_4 = \frac{\frac{1}{4}\left[\begin{array}{l}24k_2\omega_2\left(k_1-4k_2+\omega_1\right)-\\ 5\omega_2\left\{\left(k_1-4k_2+\omega_1\right)^2-16\omega_2^{\,2}\right\}\end{array}\right]}{\left(9k_2^{\,2}+25\omega_2^{\,2}\right)\left\{\begin{array}{l}\left(k_1-4k_2+\omega_1\right)^2\\ +16\omega_2^{\,2}\end{array}\right\}^2},$$

$$g_1 = \frac{\left[\begin{array}{l}\left\{k_2\left(k_1+k_2+\omega_1\right)-\omega_2^{\,2}\right\}.\\ \left(k_1+k_2+\omega_1\right)-3\omega_2^{\,2}\left(k_1+2k_2+\omega_1\right)\end{array}\right]}{\begin{array}{l}\left(k_2^{\,2}+\omega_2^{\,2}\right)\left\{\left(k_1+k_2+\omega_1\right)^2+\omega_2^{\,2}\right\}.\\ \left\{\left(k_1+k_2+\omega_1\right)^2+9\omega_2^{\,2}\right\}\end{array}},$$

$$g_2 = \frac{-\left[\begin{array}{l}3\omega_2\left\{k_2\left(k_1+k_2+\omega_1\right)-\omega_2^{\,2}\right\}\\ +\omega_2\left(k_1+2k_2+\omega_1\right)\left(k_1+k_2+\omega_1\right)\end{array}\right]}{\begin{array}{l}\left(k_2^{\,2}+\omega_2^{\,2}\right)\left\{\left(k_1+k_2+\omega_1\right)^2+\omega_2^{\,2}\right\}.\\ \left\{\left(k_1+k_2+\omega_1\right)^2+9\omega_2^{\,2}\right\}\end{array}},$$

$$g_3 = \frac{1-k_2}{k_2\left\{\left(k_1+k_2+\omega_1\right)^2+\omega_2^{\,2}\right\}},$$

$$g_4 = \frac{2}{k_2\left\{\left(k_1+k_2+\omega_1\right)^2+\omega_2^{\,2}\right\}},$$

$$f_1 = \frac{\frac{3}{4}\omega_2\left(k_1-3k_2+\omega_1\right)}{k_2\left[\left(k_1-3k_2+\omega_1\right)^2+\omega_2^{\,2}\right]^2},$$

$$f_2 = \frac{-\frac{1}{4}\left[\begin{array}{l}24k_2\omega_2\left(k_1-4k_2+\omega_1\right)-\\ 5\omega_2\left\{\left(k_1-4k_2+\omega_1\right)^2-16\omega_2^{\,2}\right\}\end{array}\right]}{\left(9k_2^{\,2}+25\omega_2^{\,2}\right)\left\{\begin{array}{l}\left(k_1-4k_2+\omega_1\right)^2\\ +16\omega_2^{\,2}\end{array}\right\}^2},$$

$$f_3 = \frac{-\frac{3}{8}\left[\left(k_1-3k_2+\omega_1\right)^2-\omega_2^{\,2}\right]}{k_2\left[\left(k_1-3k_2+\omega_1\right)^2+\omega_2^{\,2}\right]^2},$$

$$f_4 = \frac{\frac{1}{4}\left[\begin{array}{l}3k_2\left\{\left(k_1-4k_2+\omega_1\right)^2-16\omega_2^{\,2}\right\}\\ +40\omega_2^{\,2}\left(k_1-4k_2+\omega_1\right)\end{array}\right]}{\left\{\left(k_1-4k_2+\omega_1\right)^2+16\omega_2^{\,2}\right\}^2}.$$

Equation (30) has no exact solution. Since $\dot{a}$, $\dot{b}$, $\dot{\phi}$ and $\dot{\varphi}$ are proportional to the small parameter ε, therefore they are slowly varying functions of time t with the period $T = \frac{2\pi}{\omega}$. Moreover, by assuming a and b are constants in the right hand side of equation (30) and by integrating equation (30), we have

$$a = a_0 + \varepsilon \frac{3}{8} a_0 b_0^{\,2} \times \left[\begin{array}{l} \dfrac{h_1}{4\left(k_2^{\,2} + \omega_2^{\,2}\right)} \left\{ \begin{array}{l} e^{-2k_2 t} \cos\left(\begin{array}{l} 2\omega_2 t + 2\varphi \\ + \tan^{-1} \dfrac{\omega_2}{k_2} \end{array} \right) \\ - \cos\left(2\varphi + \tan^{-1} \dfrac{\omega_2}{k_2} \right) \end{array} \right\} + \\ \dfrac{h_2}{4\left(k_2^{\,2} + \omega_2^{\,2}\right)} \left\{ \begin{array}{l} e^{-2k_2 t} \sin\left(\begin{array}{l} 2\omega_2 t + 2\varphi \\ + \tan^{-1} \dfrac{\omega_2}{k_2} \end{array} \right) \\ - \sin\left(2\varphi + \tan^{-1} \dfrac{\omega_2}{k_2} \right) \end{array} \right\} + \\ \dfrac{h_3}{2k_2}\left(1 - e^{-2k_2 t}\right) + \dfrac{h_4}{2k_2} e^{-2\phi}\left(1 - e^{-2k_2 t}\right) \end{array} \right],$$

$$\phi = \phi_0 + \varepsilon \frac{3}{8} b_0^{\,2} \times \left[\begin{array}{l} \dfrac{g_1}{4\left(k_2^{\,2} + \omega_2^{\,2}\right)} \left\{ \begin{array}{l} e^{-2k_2 t} \cos\left(\begin{array}{l} 2\omega_2 t + 2\varphi \\ + \tan^{-1} \dfrac{\omega_2}{k_2} \end{array} \right) \\ - e^{-2\phi} \cos\left(\begin{array}{l} 2\varphi + \\ \tan^{-1} \dfrac{\omega_2}{k_2} \end{array} \right) \end{array} \right\} + \\ \dfrac{g_2}{4\left(k_2^{\,2} + \omega_2^{\,2}\right)} \left\{ \begin{array}{l} e^{-2(k_2 t + \phi)} \sin\left(\begin{array}{l} 2\omega_2 t + 2\varphi \\ + \tan^{-1} \dfrac{\omega_2}{k_2} \end{array} \right) \\ - e^{-2\phi} \sin\left(\begin{array}{l} 2\varphi + \\ \tan^{-1} \dfrac{\omega_2}{k_2} \end{array} \right) \end{array} \right\} \\ + \dfrac{g_3}{2k_2}\left(1 - e^{-2k_2 t}\right) + \dfrac{g_4}{2k_2} e^{-2\phi}\left(1 - e^{-2k_2 t}\right) \end{array} \right],$$

$$b = b_0 + \varepsilon b_0^{\,3} \times \left[\begin{array}{l} \dfrac{j_1 + j_2}{4\left(k_2^{\,2} + \omega_2^{\,2}\right)} \left\{ \begin{array}{l} e^{-2k_2 t} \cos\left(\begin{array}{l} 2\omega_2 t + 2\varphi \\ + \tan^{-1} \dfrac{\omega_2}{k_2} \end{array} \right) \\ - \cos\left(2\varphi + \tan^{-1} \dfrac{\omega_2}{k_2} \right) \end{array} \right\} + \\ \dfrac{j_3 + j_4}{4\left(k_2^{\,2} + \omega_2^{\,2}\right)} \left\{ \begin{array}{l} e^{-2k_2 t} \sin\left(\begin{array}{l} 2\omega_2 t + 2\varphi \\ + \tan^{-1} \dfrac{\omega_2}{k_2} \end{array} \right) \\ - \sin\left(2\varphi + \tan^{-1} \dfrac{\omega_2}{k_2} \right) \end{array} \right\} \end{array} \right],$$

$$\varphi = \varphi_0 + \varepsilon b_0^{\,2} \times \left[\begin{array}{l} \dfrac{f_1 + f_2}{4\left(k_2^{\,2} + \omega_2^{\,2}\right)} \left\{ \begin{array}{l} e^{-2k_2 t} \cos\left(\begin{array}{l} 2\omega_2 t + 2\varphi \\ + \tan^{-1} \dfrac{\omega_2}{k_2} \end{array} \right) \\ - \cos\left(2\varphi + \tan^{-1} \dfrac{\omega_2}{k_2} \right) \end{array} \right\} + \\ \dfrac{f_3 + f_4}{4\left(k_2^{\,2} + \omega_2^{\,2}\right)} \left\{ \begin{array}{l} e^{-2k_2 t} \sin\left(\begin{array}{l} 2\omega_2 t + 2\varphi \\ + \tan^{-1} \dfrac{\omega_2}{k_2} \end{array} \right) \\ - \sin\left(2\varphi + \tan^{-1} \dfrac{\omega_2}{k_2} \right) \end{array} \right\} \end{array} \right]. \quad (31)$$

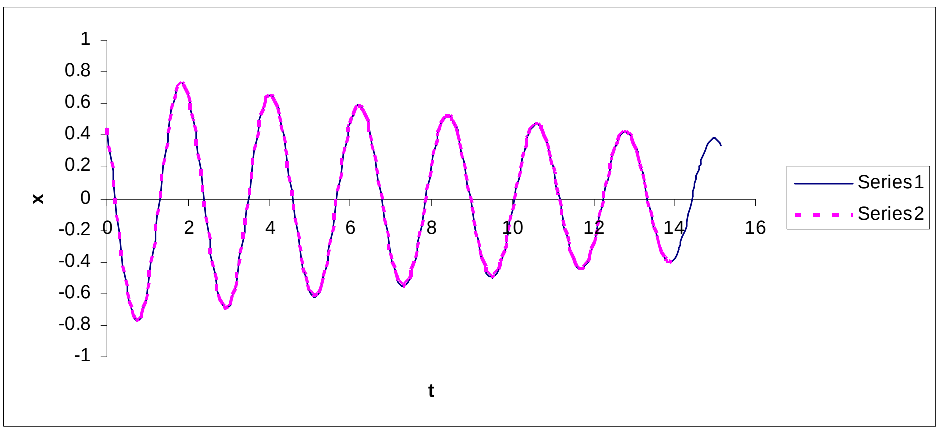
Therefore, we obtain the first order approximate solution of the equation (15) is

$$x(t,\varepsilon)=\frac{1}{2}ae^{-k_1 t}\left\{e^{-(\omega_1 t-\phi)}+te^{-(\omega_1 t+\phi)}\right\}$$
$$+be^{-k_2 t}\cos(\omega_2 t+\varphi), \qquad (32)$$

where a, b, ϕ, φ are given by the equation (31).

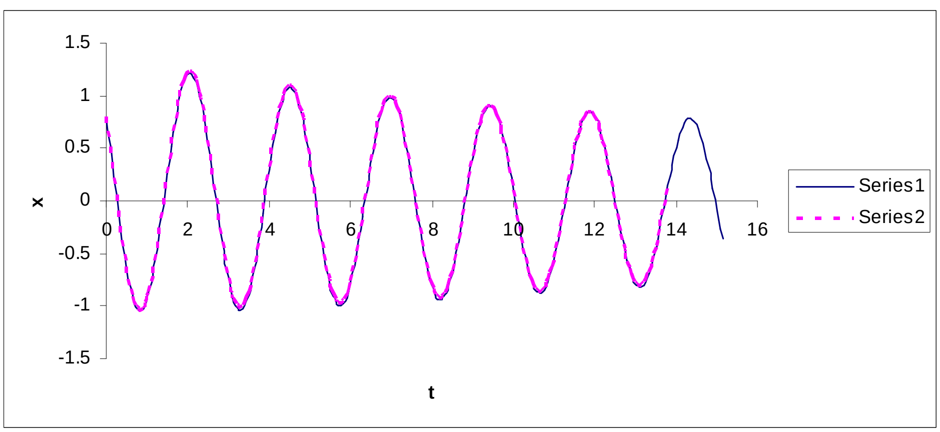
4. Results and Discussion

To test the accuracy the analytical solution obtained by the presented method of this article has been compared to the numerical solution. In this article we have computed $x(t,\varepsilon)$ by equation (32) applying the imposed conditions $\lambda_2 \approx 2\lambda_3 + 2\lambda_4$ and $\lambda_1 = \lambda_2$, in which a, b, ϕ, φ are evaluated by the equation (31). The numerical solution of (15) is computed by fourth order Runge-Kutta method, and compared with the approximate analytical solution. The approximate analytical solution and numerical solution are plotted in the figures (Fig.1 and Fig.2). From the figures we observe that our approximate analytical solution show nice coincidence with numerical solution.



$a_0 = 0.0$
$b_0 = 0.8$
$\phi_0 = 1.2$
$\varphi_0 = 0.99$
$k_1 = 0.87$
$k_2 = 0.05$
$\omega_1 = 0.1$
$\omega_2 = 2.87$
$\varepsilon = 1.0$

Fig. 1: Analytical solution in solid line — and solution by a fourth order Runge-Kutta method in dashed line - - -



$a_0 = 0.1$
$b_0 = 1.2$
$\phi(0) = 1.0$
$\varphi(0) = 1.0$
$k_1 = 0.6$
$k_2 = 0.03$
$\omega_1 = 0.25$
$\omega_2 = 2.56$
$\varepsilon = 1.0$

Fig. 2 : Analytical solution in solid line — and solution by a fourth order Runge-Kutta method in dashed line - -

5. Conclusion

To solve fourth order critically damped oscillatory nonlinear systems, the KBM method has been extended under some special conditions. When the relations $\lambda_2 \approx 2\lambda_3 + 2\lambda_4$ and $\lambda_1 = \lambda_2$ exist among the eigenvalues the method is important. The solutions obtained by this method show excellent coincidence with those obtained by numerical method.

References

1. Ali Akbar, M., Paul A. C. and Sattar M. A., An Asymptotic Method of Krylov-Bogoliubov for Fourth Order Over-damped Nonlinear Systems, Ganit, J. Bangladesh Math. Soc., Vol. **22**, pp. 83-96, 2002.
2. Ali Akbar, M., Shamsul Alam M. and Sattar M. A., Asymptotic Method for Fourth Order Damped Nonlinear Systems, Ganit, J. Bangladesh Math. Soc. Vol. **23**, pp. 41-49, 2003.
3. Ali Akbar, M., Shamsul Alam M. and M. Sattar M., A Simple Technique for Obtaining Certain Over-damped Solutions of an *n*-th Order Nonlinear Differential Equation, Soochow Journal of Mathematics Vol. **31(2)**, pp. 291-299, 2005.
4. Bogoliubov, N. N. and Mitropolskii Yu., Asymptotic Methods in the Theory of Nonlinear Oscillations, Gordan and Breach, New York, 1961.
5. Habibur Rahman, Ikramul Haque, B. M., and Ali Akbar, M., An Analytical Solution of Fourth Order Damed-Oscillatory Nonlinear Systems, Research Journal of Mathematics and Statistics, Vol.3(1), pp. 12-19, 2011.
6. Krylov, N. N. and Bogoliubov N. N., Introduction to Nonlinear Mechanics, Princeton University Press, New Jersey, 1947.
7. Murty, I. S. N., and Deekshatulu B. L., Method of Variation of Parameters for Over-Damped Nonlinear Systems, J. Control, Vol. **9**, no. 3, pp. 259-266,1969.
8. Murty, I. S. N., Deekshatulu B. L. and Krishna G., On an Asymptotic Method of Krylov-Bogoliubov for Over-damped Nonlinear Systems, J. Frank. Inst., Vol. **288**, pp. 49-65, 1969.
9. Murty, I. S. N., A Unified Krylov-Bogoliubov Method for Solving Second Order Nonlinear Systems, Int. J. Nonlinear Mech. Vol. **6**, pp. 45-53, 1971.
10. Popov, I. P., A Generalization of the Bogoliubov Asymptotic Method in the Theory of Nonlinear Oscillations (in Russian), Dokl. Akad. USSR Vol. **3**, pp. 308-310, 1956.
11. Rokibul Islam, M, Sharif Uddin M., Ali Akbar M., Azmol Huda M. and Hossain S. M. S., A New Technique for Fourth Order Critically Damped Nonlinear Systems with Some Conditions, Bull. Cal. Math. Soc., (to appear in Vol. 100(12), 2008).
12. Sattar, M. A., An asymptotic Method for Second Order Critically Damped Nonlinear Equations, J. Frank. Inst., Vol. **321**, pp. 109-113, 1986.
13. Shamsul Alam, M. and Sattar M. A., An Asymptotic Method for Third Order Critically Damped Nonlinear Equations, J. Mathematical and Physical Sciences, Vol. **30**, pp. 291-298, 1996.
14. Shamsul Alam, M. and Sattar M. A., A Unified Krylov-Bogoliubov-Mitropolskii Method for Solving Third Order Nonlinear Systems, Indian J. pure appl. Math. Vol. **28**, pp. 151-167, 1997.
15. Shamsul Alam, M., Asymptotic Methods for Second Order Over-damped and Critically Damped Nonlinear Systems, Soochow Journal of Math. Vol. **27**, pp. 187-200, 2001.
16. Shamsul Alam, M., A Unified Krylov-Bogoliubov-Mitropolskii Method for Solving *n*-th Order Nonlinear Systems, J. Frank. Inst. Vol. **339**, pp. 239-248, 2002.
17. Shamsul Alam, M., Bogoliubov's Method for Third Order Critically Damped Nonlinear Systems, Soochow J. Math. Vol. **28**, pp. 65-80, 2002.
18. Shamsul Alam, M., Method of Solution to the *n*-th Order Over-damped Nonlinear Systems Under Some Special Conditions, Bull. Cal. Math. Soc., Vol. **94**, pp. 437-440, 2002.
19. Shamsul Alam, M., On Some Special Conditions of Third Order Over-damped Nonlinear Systems, Indian J. pure appl. Math. Vol. **33**, pp. 727-742, 2002.
20. Shamsul Alam M., Asymptotic Method for Non-oscillatory Nonlinear Systems, Far East J. Appl. Math., Vol. **7**, pp. 119-128, 2002.
21. Shamsul Alam, M., On Some Special Conditions of Over-damped Nonlinear Systems, Soochow J. Math. Vol. **29**, pp. 181-190, 2003.

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