QuasiPAS Model Using The Fourth, Fifth and Sixth Positive Root of $j_2(x)$ and $j_3(x)$

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***Abstract*— QuasiPAS flow was invented by Bachtiar and James (BJ 2011). They modified Pekeris, Accad and Skholler PAS dynamo in order to get a new flow which can be fully planarized. BJ evaluated three types of quasiPAS flows: basic, partly-planarized and fully planarized. They used the first three positive roots of spherical Bessel function order two. They found nine new dynamos, unfortunately none of them is a planar velocity dynamos. In this paper, we revisited one of their flows, which is the unplanarized quasiPAS, using the fourth, fifth and sixth positive roots of spherical Bessel function order two. We found no new dynamo, but we predict that higher resolutions are needed in order to get more conclusive results.**

I. Introduction

In kinematic dynamo problem, we consider the induction equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = R\nabla \times (\mathbf{u} \times \mathbf{B}) + \nabla^2 \mathbf{B} \qquad (1)$$

Where:
B is the Magnetic field
u is the Velocity field
R (=UL/ η) is the magnetic Reynolds number

This equation shows that the growth rate of the magnetic field depends on the interaction between the magnetic field and the velocity field, and the diffusion of the magnetic field. If first term dominates, then the magnetic field will grow. In other word, the model can act as a dynamo. If the velocity is specified then the problem is known as kinematic problem. This problem is a subset of a larger problem, Magnetohydrodynamics (MHD). MHD is considered as a good model for geomagnetism because MHD has conducting fluid and shows reversal behaviour. However, MHD involves 6 equations that are needed to be solved simultaneously.

In many years, scientists found many flows that can produce dynamo. Such as, Kumar-Roberts, Pekeris-Accad-Shkollar, Dudley-James flows. But, some scientists also found some condition that is impossible to produce dynamo. One example is the Planar velocity (PVT) dynamo, which was introduced by Zel'dovich in 1957 (Zel'dovich, 1957). In his paper, Zel'dovich proved that it is impossible to have a dynamo action when the velocity is planar or parallel to a plane. However, in his proof, he assumed that the velocity occupies all space. He never provides a proof when the velocity occupies a finite volume. Moffat (1978) also mentioned, without prove, that the planar velocity also apply in a finite volume situation.

After more than 20 years, Bachtiar, Ivers and James (2006) argued that PVT is not valid for a finite volume. They showed that it is impossible to prove the theorem when the velocity field occupies a finite volume, such as a sphere. They found an example, p1Y22DM12 model, that indicates the existence of a planar velocity dynamo. However, their model, the eigenvalue, did not converge at a satisfactory level. They tried to observed the eigenvector and found that the eigenvector of p1Y22DM12 model has converged. They also try to planarized other well-known dynamo models in order to support their result. One of them is the Pekeris, Accad and Skhollar (PAS) model. PAS flow was introduced in 1973(Pekeris, Accad and Skhollar, 1973). They could not planarized the flow since the toroidal part of the flow does not satisfy the consistency condition

$$\int_0^1 r^{n+2} t_n dr = 0 \qquad (2)$$

Bachtiar and James (2011) modified PAS flow so that the new flow, quasiPAS, can be planarized. They found eight new dynamos from quasiPAS models. We revisited this flows and continue their work using the next three positive roots of spherical Bessel order two and three.

II. Mathematical Background

Bullard and Gellman (1954) give a significant contribution to the kinematic dynamo project. They propose to expand the magnetic and velocity field in toroidal and poloidal, use the spherical harmonic expansion and assume that the magnetic field is in the form **B**= **B** $e^{\lambda t}$.

$$\mathbf{B} = \sum_{n,m} (\mathbf{T}_n^m + \mathbf{S}_n^m)$$

$$\mathbf{u} = \sum_{n,m} (\mathbf{t}_n^m + \mathbf{s}_n^m)$$

where

$$\mathbf{T}_n^m = \nabla \times (rT_n^m(r,t)Y_n^m(\theta,\varphi))$$
$$\mathbf{S}_n^m = \nabla \times \nabla \times (rS_n^m(r,t)Y_n^m(\theta,\varphi))$$
$$n = 1,2,3,\ldots; m = -n,\ldots,n.$$

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$$Y_n^m = (-)^m \left[\frac{2n+1}{2-\delta_m^0}\right]^{\frac{1}{2}} P_n^m(\cos\theta)e^{im\varphi}$$
$$= (-)^m \overline{Y_n^{-m}}$$
$$Y_n^m = 0 \quad , for \quad n < |m|$$

If we discretise the radial direction using central difference scheme, we will simplify the problem into an ordinary eigenvalue problem. Our aim is to get a positive real part of the eigenvalue, λ, which indicate the growing mode of **B**. The detail of the mathematical background can be seen in Dudley and James (1989). The discussion of the spherical harmonic expansion is provided by James (1974).

III. QuasiPAS

The quasiPAS flow, which is defined as the following:

$$\mathbf{u} = 2\,Re\,\{\mathbf{s}_2^2 + \mathbf{t}_2^2\}$$

Where

$\mathbf{s}_2^2(r) = K\Lambda_i j_2(\Lambda_i r)$
$\mathbf{t}_2^2(r) = K\Lambda_i^2 j_2(\Gamma_i r)$
$\Lambda = 18.6890, 21.8538, 25.0128$
$\Gamma = 16.9236, 20.1218, 23.3042$

The toroidal part of this follow satisfies the consistency condition (2), rigid boundary and slip condition. Satisfying the consistency condition means that this flow can be transformed into a planar flow. $\boldsymbol{\Lambda}$ is the fourth, fifth and sixth positive roots of spherical Bessel function order 2. Meanwhile, $\boldsymbol{\Gamma}$ is the fourth, fifth and sixth positive roots of spherical Bessel function order 3.

Using BIJ formula, the fully-planarized quasiPAS model is defined as the following

$$\mathbf{u} = 2\mathrm{Re}\{\mathbf{s}_2^2 + \varepsilon_1\mathbf{t}_3^2 + \mathbf{t}_2^2 + \varepsilon_2(\mathbf{s}_3^2 + \mathbf{t}_4^2)\},$$

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$$s_2^2(r) = K\Lambda_i j_2(\Lambda_i r)$$
$$t_2^2(r) = K\Lambda_i^2 j_2(\Gamma_k r)$$
$$t_3^2(r) = -\alpha_3 K\Lambda_i j_3(\Lambda_i r)$$
$$s_3^2(r) = \frac{i}{2\alpha_3}\frac{\Lambda^2{}_i}{\Gamma_k} K j_3(\Gamma_k r))$$
$$t_4^2(r) = 0.5\sqrt{3}K\Lambda^2{}_i j_4(\Gamma_k r),$$

As we mentioned before, in this project, we only use the original or unplanarized quasiPAS models, i.e. $\varepsilon_1 = \varepsilon_2 = 0$. There are three kind of quasiPAS models that are needed to be elaborated in our next project.

IV. Discussion

We found no new dynamos in this project. All of the real part of the eigenvalue are negatives. However, we only investigate over the interval $0 \le R \le 2$. Beyond this range, we require more computer capacity in order to get a convergence result. To observe the problem in more detail, we also plot the radial function of the flows and get the following figures:

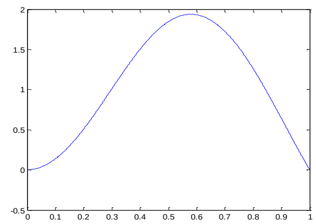
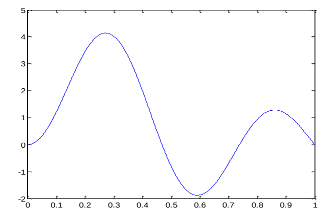
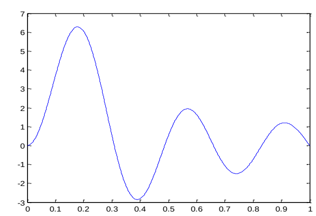
0 ≤ r ≤ 1	Graphs
s_2^2, Λ_1	
s_2^2, Λ_3	
s_2^2, Λ_4	

Fig. 1. the plot of poloidal part using three different root of $j_2(x)$.

These plots show that if we use the fourth positive root then the flow will have bigger gradient and more fluctuating. It means that the flow is more complicated than the flow with the first three positive root of $j_2(x)$. Although it is not shown, flows with fifth and sixth roots have similar pictures. As a result, they have the same problem as flow with fourth root.

In conclusion, we may need to evaluate these models using higher truncation level or using a more detail numerical methods. It is still possible that these models can act as a dynamo with higher magnetic Reynolds number.

Another approach is needed in modifying PAS model in order to search the possibility the existence of planar velocity dynamos.

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