Reparametrization and Subdivision of Interval Bezier Curves

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***Abstract*— Interval Bezier curve are new representation forms of parametric curves. Using this new representation, the problem of lack of robustness in all state-of-the art CAD systems can be largely overcome. In this paper this concept has been discussed to form a new curve over rectangular domain such that its parameter varies in an arbitrary range $[a, b]$ instead of standard parameter $[0, 1]$. Where a and b are real and, we also want that curve gets generated within the given error tolerance limit. The four fixed Kharitonov's polynomials (four fixed Bezier curves) associated with the original interval Bezier curve are obtained. A new parameterization is applied to the four fixed Kharitonov's polynomials (four fixed Bezier curves). Finally, the required interval control points are obtained from the fixed control points of the four fixed Kharitonov's polynomials. Subdividing a parametric interval Bezier curve into two interval segments is also presented. The two interval segments have the same shape as the original interval Bezier curve, but they are defined by more entities (interval control points or interval vectors) thereby making it possible to fine-tune the interval Bezier curve. Using matrix representation, it has been shown how to determine the control polygon that covers an arbitrary interval [a, b] of the original interval Bezier curve. Numerical examples are included in order to demonstrate the effectiveness of the proposed method.**

***Index Term*— Reparametrization, subdivision, interval Bezier curve, image processing, CAGD.**

I. INTRODUCTION

Computer Aided Design (CAD) is concerned with the representation and approximation of curves and surfaces, when these objects have to be processed by a computer. Parametric representations are widely used since they allow considerable flexibility for shaping and design. In Computer Aided Design and Geometric Modeling, there are considerable interests in approximating curves and surfaces with simpler forms of curves and surfaces. This problem arises whenever CAD data need to be shared across heterogeneous systems which use different proprietary data structures for model representations. For example, some systems restrict themselves to polynomial forms or limit the polynomial degree that they accommodate. Geometric modeling and computer graphics have been interesting and important subjects for many years from the point of view of scientists and engineers. One of the main and useful applications of these concepts is the treatment of curves and surfaces in terms of control points, a tool extensively used in CAGD.

There are several kinds of polynomial curves in CAGD, e.g., Bezier [1], [2], [3], [4] Said-Ball [5], Wang-Ball [6], [7], [8], B-spline curves [9] and DP curves [10], [11].

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These curves have some common and different properties.

All of them are defined in terms of the sum of product of their blending functions and control points. They are just different in their own basis polynomials. In order to compare these curves, we need to consider these properties. The common properties of these curves are control points, weights, and their number of degrees. Control points are the points that affect to the shape of the curve. Weights can be treated as the indicators to control how much each control point influences to the curve. Number of degree determines the maximum degree of polynomials. In different curves, these properties are not computed by the same method. To compare different kinds of curves we need to convert these curves into an intermediate form.

Parametric representation for curves is important in computer-aided geometric design, medical imaging, computer vision, computer graphics, shape matching, and face/object recognition. They are far better alternatives to free form representation, which are plagued with unboundedness and stability problems. Parametric representations are widely used since they allow considerable flexibility for shaping and design. A curve that actually passes through each control point is called an interpolating curve; a curve that passes near to the control points but not necessarily through them is called an approximating curve. Bezier curve is among the most commonly used method for curve and surface design, and it has been widely used in practical CAD systems.

An interval Bezier curve is a Bezier curve whose control points are rectangles (the sides of which are parallel to coordinate axis) in a plane. Such a representation of parametric curves can account for error tolerances. Based upon the interval representation of parametric curves and surfaces, robust algorithms for many geometric operations such as curve/curve intersection were proposed [12]. The series of works by the authors of [12] indicate that using interval arithmetic will substantially increase the numerical stability in geometric computations and thus enhance the robustness of current CAD/CAM systems.

Intuitively, one may consider parametrization as the journey of a particle traveling on a curve – does it move smoothly or does it face sharp turns, sudden speed and acceleration changes etc. ; and we will refer to it throughout our discussion for better assimilation of concepts. But, in the world of curve design where it may seem that we are only concerned with what a curve or surface looks like, why are we so concerned with its parametrization?

Parametrization aids computation in the sense that it provides a built-in parameter space for direct evaluation of quantities like tangents, normal, surface/ plane intersects and projections. Same curve can be represented by multiple parametrizations. Hence, in free form design reparametrization can be used to reconcile parametrization of different curve segments (or surface patches) that have been defined independently. Reparametrization of a curve means to change how a curve is parametrized, i.e., to change which parameter value is assigned to each point on the curve. Reparametrization can be performed by a parameter substitution.

In curve design, the problem often is how to balance the desire for constructing a particular shape for a curve and obtaining a proper parametrization. Most often, we may construct the initial curve to interpolate/approximate the given data points with an initial parametrization using one of the various known techniques. However, the curves are refined to achieve the desired shape by various modifications of weights and/or control points. But, our parametrization is lost! That is the small changes in curve shape might lead to a bad or improper parametrization, which if used to construct surfaces results in badly parametrized surfaces. Hence, it is necessary to reparametrize the curve/surface to correct such situations where the shape is right and the parametrization is wrong.

The problem of parametric interval Bezier curve reparametrization consists of changing the current parameter of a curve with another parameter using a reparametrization function. It should be noted that the shape of the curve remains unchanged during this process; only the way the curve is described is altered. If it is important that the degree of the curve should be kept unchanged, we may choose a linear reparametrization function.

Parametric interval Bezier curves are interactive. It is possible to control the shape of the interval Bezier curve by moving the interval control points and by smoothly connecting individual interval segments. Imagine a situation where the interval points are moved and maneuvered for a while, but the interval curve “refuses” to get the right shape. This indicates that there are not enough interval points. There are two ways to increase the number of interval points. One is to add an interval point to a segment while increasing its degree. This is called degree elevation [14], [15], [16], [17]. An alternative is to subdivide an interval Bezier curve segment into two interval segments such that there is no change in the shape of the curve [18].

This paper is organized as follows. Section II contains reparametrization of interval Bezier curve on rectangular domain, whereas section III provides subdivision of interval Bezier curves using matrix form, while section IV shows numerical examples, and the final section offers conclusions.

II. Reparametrization of Interval Bezier curve on rectangular domain

An interval polynomial is a polynomial whose coefficients are interval. We shall denote such polynomials in the form $P^I(u)$ to distinguish them from ordinary (single-valued) polynomials. In general we express an interval polynomial of degree n in the form:

$$P_n^I(u) = P_{[0,1]}^I(u) = \sum_{i=0}^{n} [p_i^-, p_i^+] \quad for\ all\ u \in [0,1] \tag{1}$$

in terms of the Bernstein polynomial basis:

$$B_i^n(u) = \binom{n}{i}(1-u)^{(n-i)}u^i\ , \quad for\ i = 0,1,\cdots,n \tag{2}$$

on $[0,1]$, where, $[p_i^-, p_i^+]$ for $(i = 0,1,\cdots,n)$ are interval control points (rectangular intervals).

Vector-valued interval $P^I(u)$ in the most general terms is defined as any compact set of points (x, y) dimensions as tensor products of scalar intervals:

$$P^I = [c_1, d_1] \times [c_2, d_2] = \{(x,y) \mid x \in [c_1,d_1] \ \ and \ \ y \in [c_2, d_2]\} \tag{3}$$

Such vector-valued intervals are clearly just rectangular regions in plane [13].

For each $u \in [0,1]$, the value $P_n^I(u)$ of the interval curve (1) is an interval vector that has the following significance: For any fixed curve $P_n(u)$ whose control points satisfy $p_i \in [p_i^-, p_i^+]$ for $(i = 0,1,\cdots,n)$ we have $P_n(u) \in P^I(u)$. Likewise, the entire interval curve (1) defines a region in the plane that contains all curves whose control points satisfy $p_i \in [p_i^-, p_i^+]$ for $(i = 0,1,\cdots,n)$.

The interval Bezier curve $P_{[0,1]}^I(u)$ with standard parameter range $u \in [0,1]$ will reparametrize as $Q_{[a,b]}^I(u)$ in the new parameter range $u \in [a,b]$. This means $Q_{[a,b]}^I(u) = \left(P_{[0,1]}^I\big((b-a)u + a\big)\right)$, where it has to satisfy the condition $Q_{[a,b]}^I(0) = P_{[0,1]}^I(a)$ and $Q_{[a,b]}^I(1) = P_{[0,1]}^I(b)$.

The four fixed Kharitonov's polynomials (four fixed Bezier curves) [19] associated with the original interval Bezier curve are:

$$\begin{aligned}
P_n^1 &= p_0^- + p_1^- u + p_2^+ u^2 + p_3^+ u^3 + p_4^- u^4 + p_5^- u^5 + \cdots \\
&\equiv \alpha_{0,n}^1 + \alpha_{1,n}^1 u + \alpha_{2,n}^1 u^2 + \cdots + \alpha_{n,n}^1 u^n \\
P_n^2 &= p_0^- + p_1^+ u + p_2^+ u^2 + p_3^- u^3 + p_4^- u^4 + p_5^+ u^5 + \cdots \\
&\equiv \alpha_{0,n}^2 + \alpha_{1,n}^2 u + \alpha_{2,n}^2 u^2 + \cdots + \alpha_{n,n}^2 u^n \\
P_n^3 &= p_0^+ + p_1^+ u + p_2^- u^2 + p_3^- u^3 + p_4^+ u^4 + p_5^+ u^5 + \cdots \\
&\equiv \alpha_{0,n}^3 + \alpha_{1,n}^3 u + \alpha_{2,n}^3 u^2 + \cdots + \alpha_{n,n}^3 u^n \\
P_n^4 &= p_0^+ + p_1^- u + p_2^- u^2 + p_3^+ u^3 + p_4^+ u^4 + p_5^- u^5 + \cdots \\
&\equiv \alpha_{0,n}^4 + \alpha_{1,n}^4 u + \alpha_{2,n}^4 u^2 + \cdots + \alpha_{n,n}^4 u^n
\end{aligned} \tag{4}$$

The four fixed Kharitonov's polynomials (four fixed Bezier curves) can be written as follows:

$$P_n^j(u) = P_{[0,1]}^j(u) = \sum_{i=0}^{n} \alpha_{i,n}^j B_i^n(u) \quad for\ all\ u \in [0,1] \ \ and \ \ (j = 1,2,3,4) \tag{5}$$

Now, the problem can be converted into: the four fixed Bezier curve $P^j_{[0,1]}(u)$ for $(j = 1,2,3,4)$ associated with the original interval Bezier curve $P^I_{[0,1]}(u)$ with standard parameter range $u \in [0,1]$ will reparametrize as $Q^j_{[a,b]}(u)$ in the new parameter range $u \in [a,b]$. Therefore, we can write:

$$Q^j_{[a,b]}(u) = P^j_n\big((b-a)u + a\big) = P^j_{[0,1]}\big((b-a)u + a\big) \quad (6)$$

or

$$\sum_{i=0}^{n} \beta^j_{i,n} B^n_i(u) = \sum_{i=0}^{n} \alpha^j_{i,n} B^n_i\big((b-a)u + a\big) \quad (j = 1,2,3,4) \quad (7)$$

where,

$$Q^j_{[a,b]}(u) = \sum_{i=0}^{n} \beta^j_{i,n} B^n_i(u) \quad (8)$$

Typically, CAGD curve constructions require use of parametric variable, u defined for a curve domain from 0 to 1 to represent curves. However, it can be transformed to another domain of parametric variable, $a \le u \le b$ by using reparameterization matrix, R. The Bezier curve which has domain of parametric variable from 0 to 1, is called "uniform Bezier curve". On the other hand, the Bezier curve with domain of parametric variable, $a \le u \le b$ can be called "non-uniform Bezier curve". For the conversions between non-uniform Bezier and CAGD curves, it is necessary to use the reparameterization matrix in order to obtain the same domains of parametric variables.

Therefore, the four fixed Kharitonov's polynomials (four fixed Bezier curves) (4) can be rewritten in the following way:

$$P^k_n(u) = \sum_{i=0}^{n}\sum_{j=0}^{n} \alpha^k_{i,n} . m_{i,j} . u^j \quad for\ all\ u \in [0,1] \quad and \quad (k = 1,2,3,4) \quad (9)$$

where,

$$m_{i,j} = (-1)^{j-i}\binom{n}{j}\binom{j}{i} \quad (10)$$

The reparameterization matrix, R can be defined as follows:

$$R = \begin{bmatrix} r_{n,n} & r_{n-1,n} & \cdots & r_{0,n} \\ r_{n,n-1} & r_{n-1,n-1} & \cdots & r_{0,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n,0} & r_{n-1,0} & \cdots & r_{0,0} \end{bmatrix} \quad (11)$$

where,

$$r_{i,j} = \binom{i}{j} a^{i-j}(b-a)^j \quad (i = 0,1,\cdots,n) \quad and \quad (j = 0,1,\cdots,n) \quad (12)$$

In CAGD curves, the reparameterization matrix can be used to transform the control points of the four fixed Kharitonov's polynomials (four fixed Bezier curves) denoted by $\{\alpha^j_{i,n}\}^n_{i=0}$ and $(j = 1,2,3,4)$, with parametric variable, $a \le u \le b$ into the control points of the corresponding four fixed Kharitonov's polynomials (four fixed Bezier curves) denoted by $\{\beta^j_{i,n}\}^n_{i=0}$, with parametric variable, $0 \le u \le 1$ as follows:

$$[\beta^j_{0,n} \ \ \beta^j_{1,n} \ \ \cdots \ \ \beta^j_{n,n}]^T = [R.M^n_B]^{-1}.M^n_B \cdot [\alpha^j_{0,n} \ \ \alpha^j_{1,n} \ \ \cdots \ \ \alpha^j_{n,n}]^T \quad (13)$$

where, M^n_B is a monomial coefficient matrix given in equation (10) as follows:

$$M^n_B = \begin{bmatrix} m_{0,n} & m_{1,n} & \cdots & m_{n,n} \\ m_{0,n-1} & m_{1,n-1} & \cdots & m_{n,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{0,0} & m_{1,0} & \cdots & m_{n,0} \end{bmatrix} \quad (14)$$

Finally, the interval control points $\{[\beta^-_i, \beta^+_i]\}^n_{i=0}$ with parametric variable, $0 \le u \le 1$, can be obtained as follows:

$$[\beta^-_i, \beta^+_i] = \big[min\big(\beta^j_i\big), max\big(\beta^j_i\big)\big] \quad (i = 0,1,\cdots,n) \quad and \quad (j = 1,2,3,4) \quad (15)$$

On the other hand, the reparameterization matrix can be used to transform the control points of the four fixed Kharitonov's polynomials (four fixed Bezier curves), denoted by $\{\tilde{\alpha}^j_{i,n}\}^n_{i=0}$ and $(j = 1,2,3,4)$, with parametric variable, $0 \le u \le 1$ into the control points of the corresponding four fixed Kharitonov's polynomials (four fixed Bezier curves), denoted by $\{\tilde{\beta}^j_{i,n}\}^n_{i=0}$, with parametric variable, $a \le u \le b$ as follows:

$$[\tilde{\beta}^j_{0,n} \ \ \tilde{\beta}^j_{1,n} \ \ \cdots \ \ \tilde{\beta}^j_{n,n}]^T = [M^n_B]^{-1}.R.M^n_B \cdot [\tilde{\alpha}^j_{0,n} \ \ \tilde{\alpha}^j_{1,n} \ \ \cdots \ \ \tilde{\alpha}^j_{n,n}]^T \quad (16)$$

and the interval control points $\{[\tilde{\beta}^-_i, \tilde{\beta}^+_i]\}^n_{i=0}$ with parametric variable, $a \le u \le b$, can be found as:

$$[\tilde{\beta}^-_i, \tilde{\beta}^+_i] = \big[min\big(\tilde{\beta}^j_i\big), max\big(\tilde{\beta}^j_i\big)\big] \quad (i = 0,1,\cdots,n) \quad and \quad (j = 1,2,3,4) \quad (17)$$

III. Subdivision of Interval Bezier Curves Using Matrix Form

An interval Bezier curve has a useful representation in a matrix form. This is a non-standard representation but extremely valuable if we can multiply matrices quickly. This form can be used to develop "subdivision matrices" that allow us to use matrix multiplication to generate different Bezier control polygons for the cubic curve.

Let $\{[p_i^-, p_i^+]\}_{i=0}^n$ be a given set of interval control points which defines the interval Bezier curve $P_n^I(u)$ for $u \in [0,1]$. In general, $[p_0^-, p_0^+] = P_n^I(0) = P_{[0,1]}^I(0)$ and $[p_n^-, p_n^+] = P_n^I(1) = P_{[0,1]}^I(1)$. For given an interval $[a, b]$ there exists a unique interval control polygon $\{[q_i^-, q_i^+]\}_{i=0}^n$ defining a Bezier curve $Q_{[a,b]}^I(u)$ such that $Q_n^I(0) = Q_{[a,b]}^I(0) = [q_0^-, q_0^+] = P_{[0,1]}^I(a)$ and $Q_n^I(1) = Q_{[a,b]}^I(1) = [q_0^-, q_0^+] = P_{[0,1]}^I(b)$.

The four fixed Kharitonov's polynomials (four fixed Bezier curves) associated with interval Bezier curve $P_n^I(u)$ with parametric variable, $0 \le u \le 1$ are:

$$P_n^j(u) = P_{[0,1]}^j(u) = \sum_{i=0}^{n} \alpha_{i,n}^j \, B_i^n(u)$$
$$for\ all\ \ u \in [0,1] \quad and \quad (j = 1,2,3,4) \tag{18}$$

The four fixed Kharitonov's polynomials (four fixed Bezier curves) associated with interval Bezier curve $Q_n^I(u)$ with parametric variable, $a \le u \le b$ are:

$$Q_n^j(u) = Q_{[a,b]}^j(u) = \sum_{i=0}^{n} \alpha_{i,n}^j \, B_i^n(u)$$
$$for\ all\ \ u \in [a,b] \quad and \quad (j = 1,2,3,4) \tag{19}$$

Defining the new four fixed Kharitonov's polynomials (four fixed Bezier curves) associated with interval Bezier curve $Q_n^I(u) = Q_{[a,b]}^I(u)$ as:

$$Q_n^j(u) = Q_{[a,b]}^j(u) = P_n^j\big((b-a)u + a\big) = P_{[0,1]}^j\big((b-a)u + a\big)$$
$$for\ all\ \ u \in [a,b]\ and\ \ (j = 1,2,3,4) \tag{20}$$

where, $Q_{[a,b]}^j(u)$ and $P_{[0,1]}^j(u)$ for $(j = 1,2,3,4)$ represent the same curves. The difference of $Q_{[a,b]}^j(u)$ and $P_{[0,1]}^j(u)$ is their parametric domains where, $P_{[0,1]}^j(u)$ for $u \in [0,1]$ and $Q_{[a,b]}^j(u) = P_{[0,1]}^j\big((b-a)u + a\big)$ for $u \in [a,b]$.

$$\begin{aligned} Q_{[a,b]}^j(u) &= P_{[0,1]}^j\big((b-a)u + a\big) \\ &= \big[\big((b-a)u + a\big)^n \quad \big((b-a)u + a\big)^{n-1} \quad \cdots \quad \big((b-a)u + a\big) \quad 1\big].M_B^n.\alpha^j \\ &= [u^n \quad u^{n-1} \quad \cdots \quad u \quad 1].C.[M_B^n]^T.\alpha^j \\ &\qquad (j = 1,2,3,4) \end{aligned}$$
$$where,$$
$$\alpha^j = \big[\alpha_{0,n}^j \quad \alpha_{1,n}^j \quad \alpha_{2,n}^j \quad \cdots \quad \alpha_{n,n}^j\big]^T \tag{21}$$

The matrix C has columns whose entries are the coefficients of $u^n, u^{n-1}, \cdots, u, 1$ in the polynomials $\big((b-a)u + a\big)^n, \big((b-a)u + a\big)^{n-1}, \cdots, \big((b-a)u + a\big), 1$ respectively, i.e.,

$$C = \begin{bmatrix} \binom{n}{n}(b-a)^n & 0 & 0 & 0 & \cdots & 0 & 0 \\ \binom{n}{n-1}a(b-a)^{n-1} & \binom{n-1}{n-1}(b-a)^{n-1} & 0 & 0 & \cdots & 0 & 0 \\ \binom{n}{n-2}a^2(b-a)^{n-2} & \binom{n-1}{n-2}a(b-a)^{n-2} & \binom{n-2}{n-2}(b-a)^{n-2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{n}{1}a^{n-1}(b-a) & \binom{n-1}{1}a^{n-2}(b-a) & \binom{n-2}{1}a^{n-3}(b-a) & \binom{n-3}{1}a^{n-4}(b-a) & \cdots & \binom{1}{1}(b-a) & 0 \\ \binom{n}{0}a^n & \binom{n-1}{0}a^{n-1} & \binom{n-2}{0}a^{n-2} & \binom{n-3}{0}a^{n-3} & \cdots & \binom{1}{0}a & 1 \end{bmatrix} \tag{22}$$

$Q_{[a,b]}^j(u)$ can be written as:

$$\begin{aligned} Q_{[a,b]}^j(u) &= [u^n \quad u^{n-1} \quad \cdots \quad u \quad 1].C.[M_B^n].\alpha^j \\ &= [u^n \quad u^{n-1} \quad \cdots \quad u \quad 1].[M_B^n]\big(S_{[a,b]}.\alpha^j\big) \\ &\qquad (j = 1,2,3,4) \end{aligned} \tag{23}$$

where,

$$S_{[a,b]} = [M_B^n]^{-1}.C.M_B^n \tag{24}$$

The new fixed control points of the four fixed Kharitonov's polynomials for the portions of the four fixed Bezier curves where u ranges from a to b can now be obtained as follows:

$$\tilde{\alpha}^j = S_{[a,b]}.\alpha^j \quad for \quad (j = 1,2,3,4) \tag{25}$$

and the required interval control points for the portion of the interval Bezier curve where u ranges from a to b can be found as:

$$[\alpha_i^-, \alpha_i^+] = \big[min\big(\tilde{\alpha}_i^j\big), max\big(\tilde{\alpha}_i^j\big)\big]$$
$$(i = 0, 1, \cdots, n) \quad and \quad (j = 1, 2, 3, 4) \tag{26}$$

IV. NUMERICAL EXAMPLES

Example 1: Consider the interval Bezier curve $P_3^I(u)$, where $0 \le u \le 1$, defined by four interval control points:

$$\begin{aligned} [p_0^-, p_0^+] &= ([60,65] \times [73,78]) \\ [p_1^-, p_1^+] &= ([40,44] \times [51,55]) \\ [p_2^-, p_2^+] &= ([22,25] \times [42,45]) \\ [p_3^-, p_3^+] &= ([16,18] \times [38,40]) \end{aligned}$$

The problem is to find an interval curve $Q_{[a,b]}^I(u)$, where $1 \le u \le 2$, i.e., $([a, b] = [1,2])$ that's defined by four interval control points $\{[\beta_i^-, \beta_i^+]\}_{i=0}^3$ such that the curve $Q_{[a,b]}^I(u)$ based on them will go from $\big(P_3^I(a) = P_{[0,1]}^I(a)\big)$ to $\big(P_3^I(b) = P_{[0,1]}^I(b)\big)$ i.e., $\big(Q_3^I(0) = Q_{[a,b]}^I(0)\big) = \big(P_3^I(a) = P_{[0,1]}^I(a)\big)$ and $\big(Q_3^I(1) = Q_{[a,b]}^I(1)\big) = \big(P_3^I(b) = P_{[0,1]}^I(b)\big)$ and will be identical in shape to $P_{[0,1]}^I(u)$ in that interval.

As explained in section II, the four fixed Kharitonov's polynomials (four fixed Bezier curves) are found, and the reparameterization matrix R and the monomial coefficient matrix M_B^3 are obtained as:

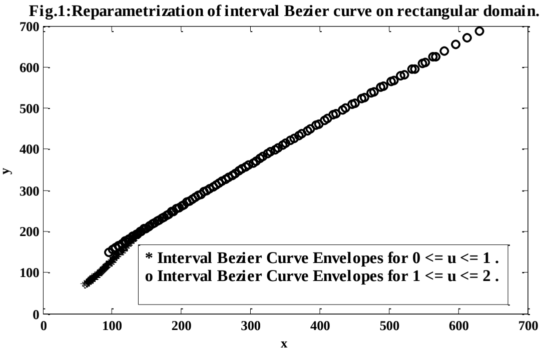
$$R=\begin{bmatrix}1&0&0&0\\3&1&0&0\\3&2&1&0\\1&1&1&1\end{bmatrix},\quad M_B^3=\begin{bmatrix}-1&3&-3&1\\3&-6&3&0\\-3&3&0&0\\1&0&0&0\end{bmatrix}$$

and the interval control points with parametric variable, $1\le u\le 2$, are obtained as follows:

$$[\beta_0^-,\beta_0^+]=([16,18]\times[38,40])$$
$$[\beta_1^-,\beta_1^+]=([7,14]\times[31,38])$$
$$[\beta_2^-,\beta_2^+]=([8,24]\times[27,43])$$
$$[\beta_3^-,\beta_3^+]=([24,63]\times[13,52])$$

Simulation results in Figure (1), shows the reparametrization of the interval Bezier curve on rectangular domain in the range [1,2].

Fig.1:Reparametrization of interval Bezier curve on rectangular domain.



Example 2: Suppose we wish to generate the interval control polygon for the portion of the curve $P_3^I(u)$, defined by four interval control points:

$$[p_0^-,p_0^+]=([1.6000,1.8500]\times[1.4000,1.7500])$$

$$[p_1^-,p_1^+]=([2.4000,2.7500]\times[3.2500,3.6500])$$

$$[p_2^-,p_2^+]=([3.9500,4.4500]\times[3.4000,3.8500])$$

$$[p_3^-,p_3^+]=([6.2500,6.7500]\times[1.8000,2.2500])$$

where, u ranges between 0 and $\frac{1}{2}$. The problem is to subdivide the curve $P_3^I(u)$ at the point $u=\frac{1}{2}$. This can be done by defining a new interval curve $Q_3^I(u)$ which is equal to $P_3^I\left(\frac{u}{2}\right)$. Clearly this new curve is a cubic polynomial, and traces out the desired portion of $P_3^I(u)$ as u ranges between 0 and 1.

As explained in section II, the four fixed Kharitonov's polynomials (four fixed Bezier curves) are found, and the matrix $S_{[a,b]}=S_{[0,\frac{1}{2}]}$ are obtained to calculate the four fixed Bezier control polygons for the first half of the curve – the portion where u ranges between 0 and $\frac{1}{2}$, as explained in section III.

$$M_B^3=\begin{bmatrix}-1&3&-3&1\\3&-6&3&0\\-3&3&0&0\\1&0&0&0\end{bmatrix},\quad C=\begin{bmatrix}\frac{1}{8}&0&0&0\\0&\frac{1}{4}&0&0\\0&0&\frac{1}{2}&0\\0&0&0&1\end{bmatrix}$$

$$S_{[0,\frac{1}{2}]}=[M_B^3]^{-1}.C.M_B^3=\begin{bmatrix}1&0&0&0\\\frac{1}{2}&\frac{1}{2}&0&0\\\frac{1}{4}&\frac{1}{2}&\frac{1}{4}&0\\\frac{1}{8}&\frac{3}{8}&\frac{3}{8}&\frac{1}{8}\end{bmatrix}$$

and the required interval control points for the first half of the curve – the portion of the interval Bezier curve where u ranges from 0 to $\frac{1}{2}$ are:

$$[\alpha_0^-,\alpha_0^+]=([1.6000,1.8500]\times[1.4000,1.7500])$$
$$[\alpha_1^-,\alpha_1^+]=([2.0000,2.3000]\times[2.3250,2.7000])$$
$$[\alpha_2^-,\alpha_2^+]=([2.6500,2.8875]\times[2.9125,3.1375])$$
$$[\alpha_3^-,\alpha_3^+]=([3.4562,3.6813]\times[2.9937,3.2125])$$

In the same way, the four fixed Bezier control polygon for the second half of the curve – the portion where u ranges between $\frac{1}{2}$ and 1.

$$S_{[\frac{1}{2},1]}=\begin{bmatrix}\frac{1}{8}&\frac{3}{8}&\frac{3}{8}&\frac{1}{8}\\0&\frac{1}{4}&\frac{1}{2}&\frac{1}{4}\\0&0&\frac{1}{2}&\frac{1}{2}\\0&0&0&1\end{bmatrix}$$

and the required interval control points for the second half of the curve – the portion of the interval Bezier curve where u ranges from $\frac{1}{2}$ to 1 are:

$$[\alpha_0^-,\alpha_0^+]=([3.4562,3.6813]\times[2.9937,3.2125])$$
$$[\alpha_1^-,\alpha_1^+]=([4.2250,4.5125]\times[3.0625,3.3000])$$
$$[\alpha_2^-,\alpha_2^+]=([5.1000,5.6000]\times[2.6000,3.0500])$$
$$[\alpha_3^-,\alpha_3^+]=([6.2500,6.7500]\times[1.8000,2.2500])$$

V. CONCLUSIONS

In this paper a new representation forms of parametric interval Bezier curves is presented. This concept has been discussed to form a new curve $Q_{[a,b]}^I(u)$ over rectangular domain such that its parameter varies in an arbitrary range $[a,b]$ where a and b are real, instead of $P^I(u)$ that its parameter varies in the range [0,1]. The problem is to find an interval Bezier curve $Q_{[a,b]}^I(u)$, where $a\le u\le b$, that's defined by interval control points $\{[\beta_i^-,\beta_i^+]\}_{i=0}^n$ such that the curve $Q_{[a,b]}^I(u)$ based on them will go from $\left(P_n^I(a)=P_{[0,1]}^I(a)\right)$ to $\left(P_n^I(b)=P_{[0,1]}^I(b)\right)$ i.e., and $\left(Q_n^I(1)=Q_{[a,b]}^I(1)\right)=\left(P_n^I(b)=P_{[0,1]}^I(b)\right)$ and will be identical in shape to $P_{[0,1]}^I(u)$ in that interval. The four fixed

Kharitonov's polynomials (four fixed Bezier curves) associated with the original interval Bezier curve are obtained. A new parameterization is applied to the four fixed Kharitonov's polynomials (four fixed Bezier curves). Finally, the required interval control points are obtained from the fixed control points of the four reparametrized Kharitonov's polynomials. Subdividing a parametric interval Bezier curve into two interval segments is also presented. The two interval segments have the same shape as the original interval Bezier curve, but they are defined by more entities (interval control points or interval vectors) thereby making it possible to fine-tune the interval Bezier curve. Using matrix representation, it has been shown how to determine the control polygon that covers an arbitrary interval $[a, b]$ of the original interval Bezier curve. Also, we saw that any change in shape of the curve often causes the initial parametrization to be lost and hence reparametrization which preserves the shape of the curve is required, which can be done in terms of certain weight relations that are preserved by reparametrization.

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